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Consistent Histories and the Interpretation of Quantum Mechanics

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The usual formula for transition probabilities in nonrelativistic quantum mechanics is generalized to yield conditional probabilities for selected sequences of events at several different times, called "consistent histories," through a criterion which ensures that, within limits which are explicitly defined within the formalism, classical rules for probabilities are satisfied. The interpretive scheme which results is applicable to closed (isolated) quantum systems, is explicitly independent of the sense of time (i.e., past and future can be interchanged), has no need for wave function "collapse," makes no reference to processes of measurement (though it can be used to analyze such processes), and can be applied to sequences of microscopic or macroscopic events, or both, as long as the mathematical condition of consistency is satisfied. When applied to appropriate macroscopic events it appears to yield the same answers as other interpretative schemes for standard quantum mechanics, though from a different point of view which avoids the conceptual difficulties which are sometimes thought to require reference to conscious observers or classical apparatus.

KEY WORDS: Joint probabilities; measurements; quantum mechanics; time reversal; wave function collapse.

1. INTRODUCTION

In this paper we introduce an extension of the standard transition probability formula of nonrelativistic quantum mechanics to certain situations, we call them "consistent histories," in which it is possible to assign joint probability distributions to events occurring at different times in a closed

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system without assuming that the corresponding quantum operators commute. The extension, which contains the usual transition probabilities as a special case, appears to be useful in throwing a new light on some of the well-known conceptual difficulties which arise in various interpretations of quantum mechanics.

In essence, nonrelativistic quantum mechanics consists in solving the Schrödinger equation and giving a physical interpretation to the solutions (including boundary and initial conditions). The former is a mathematical problem about which there is little disagreement. The latter has given rise to an extended controversy which is far from being resolved.

Most physicists accept the necessity of giving some sort of probabilistic interpretation to wave functions, and it is probabilities which in practice are compared with experimental results. However, quantum probabilities seem to differ in important respects from their classical counterparts. For example, every textbook gives the formula for calculating "the probability distribution" for the eigenvalues of any self-adjoint operator A, given a wave function or density matrix. But given two such operators which do not commute, the usual formalism gives no natural way of calculating a *joint* probability distribution of the two sets of eigenvalues. A related phenomenon is that a straightforward application of procedures which are perfectly valid for classical probabilities (e.g., as employed in classical statistical mechanics) can give wrong answers in the quantum case. Thus the probability of a quantum particle arriving at some distant point after being diffracted through a double slit cannot (in general) be calculated by first determining the probabilities that at an earlier time it passed through each of the two slits, and then the probabilities that it will reach the point in question from each of the two slits separately. Hence if probabilistic ideas are to be applied in interpreting quantum mechanics, they must either differ from classical probabilities, or their application must be restricted to special circumstances, or both.

A large and influential body of opinion, which can with some justice be considered the "mainstream" in modern physics, holds that these special circumstances are related to *measurements*: one can only speak sensibly about the probabilities of physical quantities which are, or have been, or could be measured. While a study of measurements has certainly produced significant insights into the nature of quantum mechanics, the use of "measurement" as a fundamental interpretive concept runs into difficulties. The most severe, in our opinion, is the fact that reference to measurements provides no way of interpreting what goes on in a *closed* (or isolated) quantum-mechanical system, whereas it is precisely to such a system, rather than to an open system occasionally or continuously perturbed by a quantum environment during "measurements," that the Schrödinger equation can be applied, at least in its usual form. The other difficulties which sometimes emerge from an interpretation based on measurements (the absence of invariance under time reversal, the problem of "wave function collapse," the need for a "conscious observer," etc.) are, we believe, more or less direct consequences of the problem just mentioned. (Further discussion will be found below in Section 7.)

By contrast, the consistent history approach of this paper assigns probabilities to certain sequences of events in a closed system. These sequences are selected by a mathematical consistency criterion making no reference to measurements, and can include either microscopic or macroscopic events, or both. The interpretive scheme is explicitly invariant under time reversal. Wave functions and wave function collapse are mathematical tools (frequently rather convenient ones) for obtaining answers to certain physical questions which can be equally well answered (though perhaps not as conveniently) by the use of other tools. On the other hand, there is no appeal to "hidden variables," or anything of that sort. Indeed, consistent histories can very well be thought of as an extension and (we hope) clarification of what is, by now, a "standard" approach to quantum probabilities (Section 2.3), with the latter disentangled from an unnecessary conceptual attachment to measurements which is in any case ignored by many physicists when they are actually doing quantum calculations. Measurements themselves can be studied within the consistent history framework by including the measured device along with the system being measured in a single closed quantum system, and their study from this perspective yields a number of useful insights.

The basic definitions required for the consistent history approach and formulas for the associated probabilities are given in Section 2, with some of the more technical details in Appendix A. This machinery is then applied to analyze two gedanken experiments, Sections 3 and 4, with results which are, we believe, interesting and even somewhat surprising; the reader must judge whether they are "right." Section 5 is a much more abstract discussion of a certain type of idealized measurement from the consistent history perspective. Throughout Sections 3, 4, and 5, but most explicitly in 5, we have adopted the strategy of focusing on what we feel are the features of nonrelativistic quantum mechanics which lie at the very heart of the conceptual difficulties which an interpretation must face, namely, (i) the use of (in general) noncommuting projection operators to represent physical events, and (ii) a development in time described by unitary transformations. Our goal is to make physical sense out of *this* structure, and if at times other features are ignored or subjected to a brutal oversimplification, this is done to avoid discussing items which are not central to the main conceptual issues. A brief summary of the consistent history approach along with comments on extensions and possible applications will be found in Section 6. Finally in Section 7 there is a discussion of *some* of the previous literature on quantum interpretation and its relationship to the consistent history point of view.

2. CONSISTENT HISTORIES

2.1. Events and Histories

We shall consider a closed (i.e., isolated) quantum-mechanical system S of finite but arbitrary size. The basic interpretative unit is an "event," which is to be thought of as a particular state of affairs existing at a particular time; for example, "the hydrogen atom is in its ground state," or "the needle of the meter points at 3." We assume that an event E is associated with an orthogonal projection operator, for which we use the same symbol, acting on the Hilbert space S used to describe the system of interest. Eigenvectors of E with eigenvalue 1 are to be thought of as states of the system for which this event occurs (or "exists"), those with eigenvalue 0 as states for which it does not occur, or for which the event E' standing for "not E" occurs; the two projections are related by

$$E' = 1 - E \tag{2.1}$$

A "history" $\mathscr{H}$ of S is a sequence of events

$$D \to E_1 \to E_2 \to \cdots \to E_n \to F \tag{2.2}$$

occurring at a set of times

$$t_0 < t_1 < t_2 \cdots < t_n < t_f \tag{2.3}$$

where t_0 is the time of the initial event D, and t_f that of the final event F. The strict inequalities in (2.3) can be replaced by $\leqslant$ provided *no two noncommuting projection operators are assigned to the same time*. (It is sometimes convenient to use a density matrix in place of a projection operator for D or F, but we shall not do so in this paper.)

Rather than an individual event, it will frequently be convenient to consider an *event set* $[E_k^\alpha]$ at the time t_k. By this we shall mean that there is a decomposition of the identity operator,

$$1 = \sum_{\alpha=1}^{M_k} E_k^\alpha \tag{2.4}$$

in terms of orthogonal projections satisfying

$$E_k^\alpha E_k^\beta = \delta_{\alpha\beta} E_k^\alpha \tag{2.5}$$

(Note that the superscript is a label, not an exponent.) Then the set $[E_k^\alpha]$ consists of each of the E_k^α along with all sums of two or more *distinct* E_k^α, including the identity itself; thus a total of $2^{M_k} - 1$ projections. The simplest situation is that in which $M_k = 2$, in which case the event set consists of E, E', and 1 in the notation (2.1). The *family* $\mathscr{F}$ of histories

$$D \rightarrow [E_1^\alpha] \rightarrow [E_2^\alpha] \rightarrow \cdots [E_n^\alpha] \rightarrow F \tag{2.6}$$

consists of all histories $\mathscr{H}$ of the form (2.2), where E_j is a member of the event set $[E_j^\alpha]$.

2.2. Weights, Consistency, and Probabilities

We assume that the time development of S is governed by a unitary transformation $U(t', t)$ which maps states at time t into those at time t'. We make the usual assumptions that U is continuous in both arguments and that it satisfies

$$U(t'', t')U(t', t) = U(t'', t) \tag{2.7}$$

and

$$U(t, t) = 1 \tag{2.8}$$

Thus in particular $U(t, t')$ is the inverse of $U(t', t)$. In addition to the "Schrödinger" operators in (2.2) it is convenient to define the corresponding "Heisenberg" operators

$$\hat{E}_j = U(t_r, t_j) E_j U(t_j, t_r) \tag{2.9}$$

referred to a particular *reference time* t_r which is independent of j. The same formula defines $\hat{D}$ and $\hat{F}$ (with t_j replaced by t_0 and t_f, respectively) in terms of D and F.

The weight w associated with a history (2.2) is defined by (Tr = trace)

$$\begin{aligned} &w(D \wedge E_1 \wedge E_2 \wedge \cdots E_n \wedge F) \\ &\quad = \mathrm{Tr}[\hat{E}_n \hat{E}_{n-1} \ldots \hat{E}_2 \hat{E}_1 \hat{D} \hat{E}_1 \hat{E}_2 \ldots \hat{E}_{n-1} \hat{E}_n \hat{F}] \end{aligned} \tag{2.10}$$

where "$\wedge$" should be read as "and." The right side does not depend on t_r, all references to which may be eliminated by inserting the specific time transformations:

$$\begin{aligned} &w(D \wedge E_1 \wedge \cdots E_n \wedge F) \\ &\quad = \mathrm{Tr}[U(t_f, t_n) E_n U(t_n, t_{n-1}) \cdots \\ &\quad\quad \times U(t_2, t_1) E_1 U(t_1, t_0) D U(t_0, t_1) E_1 U(t_1, t_2) \cdots \\ &\quad\quad \times U(t_{n-1}, t_n) E_n U(t_n, t_f) F] \end{aligned} \tag{2.11}$$

In the case of an infinite-dimensional Hilbert space we shall always assume that the trace of either D or F is finite, so these equations make sense.

Next define the conditional weight W by the formula

$$W(E_1 \wedge E_2 \wedge \ldots E_n \mid D \wedge F) = w(D \wedge E_1 \wedge \cdots E_n \wedge F)/w(D \wedge F) \tag{2.12}$$

provided

$$w(D \wedge F) = \mathrm{Tr}[\hat{D}\hat{F}] = \mathrm{Tr}[U(t_f, t_0) D U(t_0, t_f) F] \tag{2.13}$$

does not vanish. In certain respects $W(A \mid B)$, the "weight of A given B," behaves like a conditional probability of A given B (e.g., it is real and nonnegative), but in other respects it does not (e.g., it can be greater than 1). To get around this difficulty, which is a quantum effect in the sense that it arises from the noncommutativity of the operators in (2.10), we introduce the concept of a *consistent family*. The family $\mathscr{F}$ of histories (2.6) will be called consistent when for *every* k, $1 \leqslant k \leqslant n$, and every history $\mathscr{H}$ in $\mathscr{F}$ it is the case that

$$\begin{aligned} &w(D \wedge E_1 \wedge \cdots E_k \wedge \cdots E_n \wedge F) \\ &\quad = \sum_\alpha{}' w(D \wedge E \wedge \cdots E_k^\alpha \wedge \cdots E_n \wedge F) \end{aligned} \tag{2.14}$$

where $\sum'$ on the right side means a sum over precisely those projections which make up E_k on the left:

$$E_k = \sum_\alpha{}' E_k^\alpha \tag{2.15}$$

When (and only when) these consistency conditions are satisfied, we shall refer to the weights W appearing in (2.12) *as "probabilities," and replace the symbol $W(A \mid B)$ by $P(A \mid B)$, the probability of A* given B. This terminology is appropriate, as shown in Appendix A, when the events in question are all associated with a single consistent family, and in this case the formulas can be manipulated as in classical probability theory. For example,

$$P(E_1 \mid D \wedge E_2 \wedge F) = P(E_1 \wedge E_2 \mid D \wedge F)/P(E_2 \mid D \wedge F) \tag{2.16}$$

$$P(E_2 \mid D \wedge F) = \sum_\alpha P(E_1^\alpha \wedge E_2 \mid D \wedge F) \tag{2.17}$$

are correct formulas provided

$$D \to [E_1^\alpha] \to [E_2^\alpha] \to F \tag{2.18}$$

is a consistent family. [The left side of (2.16) is, of course, only defined when $P(E_2 \mid D \wedge F)$ is positive; in what follows we shall not always take the trouble to make an explicit qualification of this sort.] In Appendix A it is also shown that (2.14) is equivalent to

$$\operatorname{Re}\operatorname{Tr}\left[\hat{E}_n \ldots \hat{E}_k^\alpha \ldots \hat{E}_1 \hat{D} \hat{E}_1 \ldots \hat{E}_k^\beta \ldots \hat{E}_n \hat{F}\right] = 0 \tag{2.19}$$

for every pair $\alpha < \beta$, where Re denotes the real part.

In the arguments of $P(\ \mid\ )$, though not that of $w(\ )$, we shall feel free to write the events in other than the temporal order, when that is convenient. The symbol $\mathscr{H}$ will be used for the string $D \wedge E_1 \wedge \cdots E_n \wedge F$ as well as for (2.2), and in the case where consistency is fulfilled the left side of (2.12) can be written as $P(\mathscr{H} \mid D \wedge F)$, since $P(A \wedge B \mid B)$ is the same as $P(A \mid B)$. However, it is to be understood that there is a definite temporal order associated with a history, and weights of the form (2.12) are always *defined* with the operators inside the trace in (2.10) in the appropriate temporal order. (If some of the operators commute, it may be possible to *calculate* the weights with the operators in a different order and get the same answer.)

To avoid confusion, we note that (2.12) or a very similar formula occurs in the "orthodox" interpretation of quantum mechanics for sequences of events which do not satisfy any consistency condition. However, its physical interpretation is in general quite different from that appropriate for (2.12) applied to a consistent history. See the discussion related to Equation (7.1) in Section 7.1.

A *particular* history of the form (2.2) will be called *consistent* provided the smallest family of histories in which it can be embedded is consistent; otherwise it is inconsistent. This smallest family is that in which for each k the decomposition (2.4) is of the form $1 = E_k + E_k'$. It is important to notice that if a history $\mathscr{H}$ is consistent, the corresponding conditional probability (2.12), $P(\mathscr{H} \mid D \wedge F)$, does *not* depend on the family of consistent histories in which $\mathscr{H}$ happens to be embedded for a particular discussion; this is obvious from the fact that the right side of (2.12) depends only on the events which make up $\mathscr{H}$. The consistency of a particular history (2.2) can be checked using (2.19) in the following form. Let $\hat{G}_j$ denote $\hat{E}_j$ or $\hat{E}_j'$ or 1. Then consistency is equivalent to the demand that for every k between 1 and n, and every possible choice for the $\hat{G}_j$ with $j \neq k$,

$$\operatorname{Re}\operatorname{Tr}\left[\hat{G}_n \ldots \hat{E}_k \ldots \hat{G}_1 \hat{D} \hat{G}_1 \ldots \hat{E}_k' \ldots \hat{G}_n \hat{F}\right] = 0 \tag{2.20}$$

The final event F was introduced as one of the conditions in (2.12) partly as a matter of convenience and partly to demonstrate that this basic interpretive formula along with the consistency conditions is invariant under time reversal in the sense that the roles of D and F can be formally interchanged provided the order of events is reversed. Thus the invariance of the trace under cyclic permutation means that (2.10) is the same as

$$\operatorname{Tr}\left[\hat{E}_1 \hat{E}_2 \ldots \hat{E}_n \hat{F} \hat{E}_n \ldots \hat{E}_1 \hat{D}\right] \tag{2.21}$$

It is of course possible to study histories in which the final "event" is the dummy $F = 1$ (or in which the initial $D = 1$). In this case we omit F on the right side of the vertical bar and write $W(\ldots \mid D)$, or $P(\ldots \mid D)$ in the consistent case. Note, however, that in the consistency conditions (2.14) the events D and F play a distinguished role. Thus, the consistency of (2.2) in the case $F = 1$ implies the consistency of the history

$$D \to E_1 \to E_2 \to \cdots \to E_{n-1} \to E_n \tag{2.22}$$

in which E_n plays the role of the "final event," but the consistency of (2.22) does not necessarily imply that of (2.2) with $F = 1$.

The consistency of a particular history or family depends, in general, upon the choice of all of the events (*including* D and F) or event sets. Consequently it is usually not possible to add a new event set to a consistent family and still satisfy the consistency conditions. When, however, this is possible, we shall say that the family and the new event set are *compatible* (and, in the contrary case, that they are are *incompatible*). To be more precise, a family $\mathscr{F}$ and an event set $[A_\mu^\alpha]$ at time t_μ are compatible provided there exists a consistent family $\mathscr{F}'$ with the same D and F as $\mathscr{F}$ which includes $[A_\mu^\alpha]$ and all the event sets of $\mathscr{F}$. This means, in particular, that t_μ must fall between t_0 and t_f, and t_μ can coincide with one of the times associated with an event set in $\mathscr{F}$ only if all the projections in $[A_\mu^\alpha]$ commute with all of those in the corresponding event set in $\mathscr{F}$; see the remarks following (2.3). In addition, $\mathscr{F}$ may be compatible with $[A_\mu^\alpha]$ at time t_μ and also with $[B_\mu^\alpha]$ at time t_ν without there being a consistent family which includes $[A_\mu^\alpha]$, $[B_\nu^\alpha]$ and all the event sets of $\mathscr{F}$. When, however, such a family does exist we shall say that $\mathscr{F}$ is compatible with $[A_\mu^\alpha]$ and $[B_\nu^\alpha]$ *together*.

A similar terminology can be applied to a single history: A consistent history $\mathscr{H}$ is *compatible* with the event set $[A_\mu^\alpha]$ if the smallest family containing $\mathscr{H}$ is compatible with $[A_\mu^\alpha]$, and $\mathscr{H}$ is compatible with the event A_μ provided this smallest family is compatible with the set $\{A_\mu, A'_\mu, 1\}$. Comparable definitions apply for "$\mathscr{H}$ is compatible with A_μ and B_ν together," etc.

The formal definition of consistency does not provide much physical insight into its meaning. The specific examples of Sections 3 and 4 will be helpful in supplying a certain amount of intuition. At this point it may be useful to remark that since the quantum phenomena which are troublesome from the point of view of classical probabilities typically arise because of the "interference" of quantum amplitudes, the consistency conditions (2.14) may be understood intuitively as the assertion that such interference effects are negligible *so far as the events in* $\mathscr{F}$ are concerned. Similarly, (2.19) may be interpreted as the assertion that there is no interference in the quantum amplitudes propagating from D to F along the separate paths passing through E_k^α and E_k^β; the significance of this will become clearer from the detailed argument beginning at (3.7) applied to the example of Section 3.

Just as in optics one finds situations in which interference effects, always present in principle, can be ignored in practice, similarly in applications of the consistency conditions there are cases in which small violations occur—the probabilities do not quite "add up"—but can be ignored. See the remarks in Section 3.3 below for a specific example. Given that nonrelativistic quantum mechanics cannot, in any case, be ultimately precise, and that there is typically a certain amount of ambiguity in defining projection operators associated with macroscopic events, there seems to be little point in worrying about violations which are small in some appropriate sense, though obviously this is a matter which deserves further study.

2.3. The Standard Statistical Interpretation

In the case of the family

$$D \to [E_1^\alpha] \to 1 \tag{2.23}$$

the consistency conditions are automatically satisfied, and the interpretation of W in (2.12) as a probability (conditional on the initial state D) is an accepted part of the usual quantum interpretation. In particular, if D and E_1 correspond to the (normalized) wave functions φ and ψ, (2.12) yields the well-known transition probability

$$P(E_1 \mid D) = W(E_1 \mid D) = |\langle\psi| U(t_1, t_0)|\varphi\rangle|^2 \tag{2.24}$$

We thus feel justified in referring to (2.12) applied to (2.23) as the "standard statistical interpretation of quantum mechanics," or "standard interpretation" for short. A slight generalization is to the case in which the Heisenberg operators at the common reference time associated with the different event sets $[E_1^\alpha] \ldots [E_n^\alpha]$ *and* F in (2.6) all commute with each other:

$$\begin{aligned} \hat{E}_j^\alpha \hat{E}_k^\beta &= \hat{E}_k^\beta \hat{E}_j^\alpha \\ \hat{E}_j^\alpha \hat{F} &= \hat{F} \hat{E}_j^\alpha \end{aligned} \tag{2.25}$$

for all j and k ($\geqslant 1$) and all α and β. In this situation, as also in the case where $\hat{D}$ instead of $\hat{F}$ commutes with all the $\hat{E}_j^\alpha$, the consistency condition is again automatically satisfied, since permutation of operators inside the trace along with $(\hat{E}_j)^2 = \hat{E}_j$ can be used to eliminate the "extra set" of $\hat{E}$'s preceding $\hat{D}$ on the right side of (2.10). We shall call (2.12) for this case the "generalized standard interpretation." Thus the interesting cases of consistent histories which go beyond the "standard interpretation" are those in which the projection $\hat{E}$ corresponding to at least one event fails to commute with the other operators in such a way that it must still occur twice in the trace in (2.10).

3. SCATTERING INTO TWO COUNTERS

3.1. The Gedanken Experiment

As a first application of the consistent history approach consider the scattering problem of Fig. 1. A wave packet ψ_0 represents a particle at time t_0 traveling toward a scattering center (regarded as a fixed potential) from which it emerges at a later time t_1 in the form of a superposition of two packets ψ_1^a and ψ_1^b, the first traveling toward a counter C_a and the second toward a counter C_b. The counters are constructed to register the passage of a particle without significantly perturbing its motion, and at a later time t_2 the particle has passed through one or the other counter, resulting in the wave packets ψ_2^a and ψ_2^b. [In place of the scattering problem one could equally well imagine the decay of an unstable system, located at the

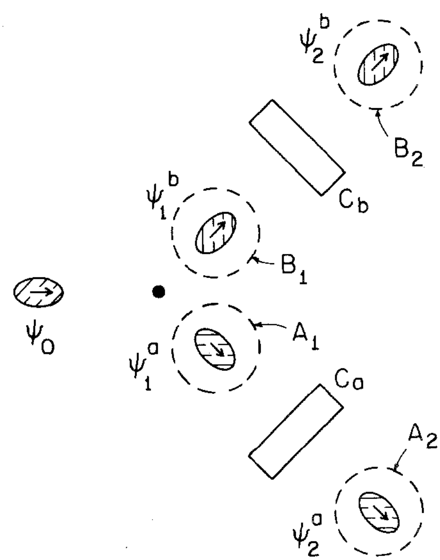

Fig. 1. Idealized scattering experiment. The scattering center is the large dot and the counters are represented by rectangles. The particle wave packets are shown schematically by the cross hatched ovals. The regions A_1, B_1, etc. are explained in the text.

scattering center, with the emission of a *single* particle represented by the packet $(\psi_1^a + \psi_1^b)/\sqrt{2}$ at time t_1.]

As the consistent history approach is to be applied to a closed system, it is necessary to think of the particle plus the two counters as constituting a single quantum mechanical system. We suppose that at time t_0 the initial state D is the projection operator $|\Psi_0\rangle\langle\Psi_0|$, where Ψ_0 is the wave function

$$t = t_0: \quad \Psi_0 = \psi_0 C_a C_b \tag{3.1}$$

The time development given by the Schrödinger equation transforms (3.1) into

$$t = t_1: \quad \Psi_1 = (\psi_1^a + \psi_1^b) C_a C_b/\sqrt{2} \tag{3.2}$$

$$t = t_2: \quad \Psi_2 = (\psi_2^a C_a^+ C_b + \psi_2^b C_a C_b^+)/\sqrt{2} \tag{3.3}$$

at these later times. Here C_a stands for the wave function of this counter in an "untriggered" state, while C_a^+ is the wave function for a "triggered" state resulting from the passage of a particle through the counter; all other changes in the counter wave function, such as that which occurs between t_0 and t_1, are ignored in this notation. A parallel notation is employed for counter b. Note that (3.3) is a plausible consequence of (3.2) inasmuch as $\psi_1^a C_a C_b$ should become $\psi_2^a C_a^+ C_b$ at a later time: a particle passing through counter C_a triggers it without affecting counter C_b, while one passing through C_b does not affect C_a. In what follows it will be convenient to suppose that the individual functions ψ_1^a, C_a, C_b^+, etc. appearing on the right side of (3.1) to (3.3) are all normalized (norm 1). In (3.3) Ψ_2 is a typical example of what we shall call a "grotesque" wave function formed from the superposition of states corresponding to distinct macroscopic situations.

3.2. Answers to Two Questions

We shall now apply the consistent history approach in order to answer the following questions. Given the initial state D (corresponding to the wave function Ψ_0):

1. If at time t_2 counter a is in the triggered state C_a^+, what is the probability that at the earlier time t_1 the particle was in the region A_1 (Figure 1)?

2. If at time t_2 counter a is in the triggered state C_a^+, what is the probability that at the same time the particle is in the region A_2?

The projection operator corresponding to the particle being in the region A_j at time t_j, which we shall also denote by A_j, corresponds to multiplying the particle portion of the wave function by a function which is

1 inside the region in question, and 0 outside. Thus, for example,

$$\begin{aligned} A_1\Psi_1 &= \Psi_1^a/\sqrt{2} = \psi_1^a C_a C_b/\sqrt{2} \\ A_1'\Psi_1 &= \Psi_1^b/\sqrt{2} = \psi_1^b C_a C_b/\sqrt{2} \end{aligned} \tag{3.4}$$

(remember that $A_1' = 1 - A_1$). We also need the projection operator K_a^+ corresponding to the event (which we denote by the same symbol) that counter a is in the triggered state C_a^+ at time t_2. It has the property that

$$K_a^+ C_a^+ = C_a^+, \qquad K_a^+ C_a = 0 \tag{3.5}$$

The first of the two questions given above can be answered within the consistent history interpretation by calculating the conditional probability $P(A_1 \mid D \wedge K_a^+)$ using (2.12), with $F = K_a^+$, provided the corresponding history

$$D \to A_1 \to K_a^+ \tag{3.6}$$

is consistent. To check consistency, we must show that

$$\operatorname{Re}\operatorname{Tr}\left[\, U(t_2, t_1)A_1 U(t_1, t_0) D U(t_0, t_1) A_1' U(t_1, t_2) K_a^+ \,\right] = 0 \tag{3.7}$$

where (2.20) has been written in the "long" form corresponding to (2.11).

To begin with, note that

$$U(t_1, t_0) D U(t_0, t_1) = |\Psi_1\rangle\langle\Psi_1| = D_1 \tag{3.8}$$

in dyadic notation, where Ψ_1 is given by (3.2), and D_1 is a convenient shorthand for the corresponding projection. Hence, using (3.4),

$$A_1 D_1 A_1' = \tfrac{1}{2}|\Psi_1^a\rangle\langle\Psi_1^b| \tag{3.9}$$

and consequently we have

$$U(t_2, t_1) A_1 D_1 A_1' U(t_1, t_2) = \tfrac{1}{2}|\Psi_2^a\rangle\langle\Psi_2^b| \tag{3.10}$$

where

$$\Psi_2^a = \psi_2^a C_a^+ C_b, \qquad \Psi_2^b = \psi_2^b C_a C_b^+ \tag{3.11}$$

Thus the trace in (3.7) is equal to

$$\tfrac{1}{2}\langle\Psi_2^b|K_a^+|\Psi_2^a\rangle = 0 \tag{3.12}$$

since $K_a^+\Psi_2^b = 0$ [see (3.5)], and hence (3.7) is satisfied.

Having checked consistency, we can use (2.12) to calculate the desired probability. The numerator can be computed using the same steps employed to check (3.7), with A_1' replaced by A_1 and Ψ_j^b by Ψ_j^a. The final trace is equal to

$$\tfrac{1}{2}\langle\Psi_2^a|K_a^+|\Psi_2^a\rangle = \tfrac{1}{2} \tag{3.13}$$

The denominator in (2.12) is given by

$$\langle\Psi_2|K_a^+|\Psi_2\rangle = \tfrac{1}{2} \tag{3.14}$$

whence it follows, dividing (3.13) by (3.14), that

$$P(A_1 \mid D \wedge K_a^+) = 1 \tag{3.15}$$

A completely analogous calculation shows that $D \to B_1 \to K_A^+$ is consistent and that

$$P(B_1 \mid D \wedge K_a^+) = 0 \tag{3.16}$$

In words, (3.15) and (3.16) mean that given that counter a is in the triggered state at time t_2, we can be certain that the particle was in region A_1 at the earlier time t_1, and that it was not in region B_1.

To answer the second question we need to compute $P(A_2 \mid D \wedge K_a^+)$. Since A_2 and K_a^+ refer to atomic positions in different spatial regions at the same time (K_a^+ can be thought of as associated with the position of a pointer which is part of counter a), these two projections commute, and thus (Section 2.3) the history $D \to A_2 \to K_a^+$ is automatically consistent. A straightforward calculation using (2.6) then yields the result

$$P(A_2 \mid D \wedge K_a^+) = 1 \tag{3.17}$$

and by similar reasoning we obtain

$$P(B_2 \mid D \wedge K_a^+) = 0 \tag{3.18}$$

In words, if counter a is in the triggered state at time t_2, then the particle is certainly in region A_2 and definitely not in region B_2 at this time.

3.3. Comments on the Calculations of Section 3.2

It is encouraging that the consistent history approach yields "physically obvious" answers to questions 1 and 2 of the previous section, and it is worthwhile examining in some detail how it does so. We shall concentrate on question No. 1, as it is more interesting from the view of quantum interpretation, and add some remarks on question No. 2 at the end.

First it is worth noting that the consistency check for the history (3.6) is nontrivial, since $\hat{A}_1$ does not commute with $\hat{D}$ or with $\hat{K}_a^+$ (see the argument in Appendix B). This means that in order to answer question 1 within a probabilistic framework applied to a closed system, it is necessary to go beyond the standard probabilistic interpretation of quantum mechanics (Section 2.3). Consistent histories represent one such extension: there may well be others. But in any case *some* extension is needed if one is to answer question 1. Second, note that in the consistency check, in particular in writing (3.4), we implicitly used the fact, suggested by the sketch in Figure 1, that the wave packet ψ_1^a vanishes outside the region A_1. It might be more realistic to assume that instead of vanishing, ψ_1^a has a very small amplitude outside A_1. In that case one could expect to find a contribution to (3.7) arising, in effect, from an interference between the part of ψ_1^a outside and the part inside A_1 as they propagate forward in time, and thus a small violation of the consistency condition. This possibility has already been noted in Section 2.2, and the present example may help to indicate the sort of situation in which small ("negligible") violations of consistency could be tolerated without too great a concern about the physical interpretation.

Next note that the consistent history approach provides an answer to question 1 without "collapsing wave functions": the answer is obtained from a conditional probability which results from evaluating the traces of the type (2.11). It is nonetheless of interest to note that the procedure we employed for evaluating the numerator in (2.12) involved an operation which in many ways resembles the collapse of a wave function. Namely, in going from D_1 to $A_1D_1A_1$ [see (3.8) and (3.9) for the corresponding steps in the consistency check] one in effect replaces Ψ_1 by the "collapsed" function $\Psi_1^a\sqrt{2}$ [see (3.4)]. Indeed, if D is a pure state, one can always visualize the process of calculating (2.11) as a series of "collapses" occurring at successively later times.

What this indicates is that consistent histories have at least some *formal* correspondence with interpretations of quantum mechanics involving wave function "collapse" in one form or another (including the case in which the wave function is replaced by a density matrix) (see Section 7). However, the example under discussion also indicates important differences. The collapse employed in calculating the numerator in (2.12) obviously has nothing to do with measurements (though it can be connected with a sort of ideal "measurability"; see Section 5), since it occurs at a time t_1 *before* the particle has interacted with either of the counters. It in fact "occurs" as a result of using the mathematical apparatus of consistent histories to answer a particular physical question; given a *different* physical question, one would not in general carry out this particular "collapse."

Despite the fact that the consistent approach is applied to a closed system, *at no point in answering question* 1 (or question 2) *is it necessary to prove an interpretation for the grotesque wave function* Ψ_2 in (3.3). The reader may object that while this function is not "obviously" present in the formal apparatus of (2.12), it is implicitly present in that we have actually employed it in evaluating the denominator in (2.12); see (3.14). True enough, but this was certainly not necessary. The fact that the interpretive machinery of the consistent history approach is independent of the sense of time, as noted in Section 2.2, means that in place of (3.14) we could equally well have evaluated $\langle\Psi_0|\hat{K}_a^+|\Psi_0\rangle$, with

$$\hat{K}_a^+ = U(t_0, t_2)K_a^+U(t_2, t_0) \tag{3.19}$$

the projection corresponding to K_a^+ at the reference time t_0. This route dispenses of all mention of the grotesque wave function Ψ_2, to be sure, at the price of introducing an equally or even more grotesque operator $\hat{K}_a^+$. However, the point we wish to make is that within the consistent histories interpretive scheme there is no difference in principle between the two time directions, and for this reason, among others, it is rather unnatural to regard Ψ_2 as somehow the unique "wave function of the universe" at $t = t_2$.

Nothing in the previous remarks rules out the possibility of introducing the "grotesque" *event*

$$G = |\Psi_2\rangle\langle\Psi_2| \tag{3.20}$$

at time t_2 and incorporating it into some consistent history. Such histories do exist, and the consistent history approach will assign probabilities to consistent families of "grotesque" histories if that is what interests the theoretician. The point we wish to make is not that grotesque events are somehow ruled out by the consistent history approach (obviously they are not), but simply that they are not an essential part of interpreting what happens in an "ordinary" consistent history.

Some further remarks may help in clarifying the role of wave functions in the consistent history approach. Any event associated with an operator E_j projecting onto a one-dimensional subspace can be associated with the corresponding unique wave function (apart from phase and normalization, which are not the point at issue), and if this event occurs at time t_j, it is quite proper from the consistent history perspective to say that the system is completely described by this wave function *at this time*. (In particular, there are no "hidden variables.") But even when this event does occur, the consistent history interpretation does not ascribe any corresponding unique physical significance to the Heisenberg operator $\hat{E}_j$ (2.9) at times t_r earlier than or later than t_j. True enough, such operators enter into expressions such as (2.10), but here the time t_r is arbitrary, and if the system is somehow described at time t_r by one of these operators, the question is: which one? To speak of a unique "wave function of the system," in the sense of Ψ_2 at time t_2, corresponds to giving a privileged role to $\hat{D}$ at all times, and this seems neither necessary nor very natural from the consistent history point of view.

As to question No. 2, the main point of interest is that it can be easily answered by the alternative route of "collapsing" the wave function Ψ_2 in the sense of replacing the projection operator $|\Psi_2\rangle\langle\Psi_2|$ by the corresponding density matrix [see the definition in (3.11)]

$$\tfrac{1}{2}|\Psi_2^a\rangle\langle\Psi_2^a| + \tfrac{1}{2}|\Psi_2^b\rangle\langle\Psi_2^b| \tag{3.21}$$

using some justification such as that discussed in Section 7.2.1. [Presumably this collapse should be carried out just after the particle has finished interacting with (one of) the counters, but the result will in any case be the same at time t_2.] While this procedure works quite well for question No. 2, it is not obvious how to apply it in order to answer question No. 1. We shall have more to say about it, in the context of a slightly different problem, in Section 4.3 below.

4. SPIN POLARIZATION MEASUREMENTS

4.1. The Gedanken Experiment

The problem of interest is the passage of a spin-1/2 particle through two successive spin polarization analyzers. Such an analyzer (Fig. 2) has three parts. First there is a region R_1 with a magnetic field gradient arranged so that a beam of particles entering from the left, along the $+y$ axis, will be separated into two beams depending on whether the z component of spin S_z is $1/2$ or $-1/2$ (in units of $\hbar$). Next there is a pair of counters, one in each beam, which register the passage of a particle while producing a negligible perturbation of its motion and without disturbing its spin polarization. Finally another region R_2 of magnetic field gradient recombines the beams from the two counters into a single beam emerging on the right. We assume in particular that the construction is such that if a

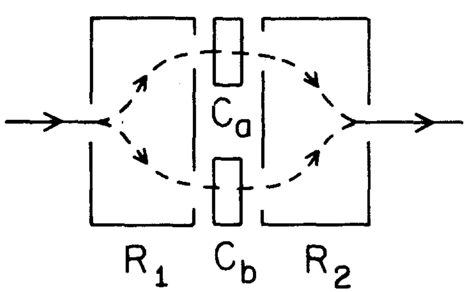

Fig. 2. Schematic drawing of a spin polarization analyzer. The regions R_1 and R_2 of nonuniform magnetic field split and then reunite beams of particles with opposite spin polarization, while C_a and C_b are counters detecting particles passing through the separated beams.

particle with $S_z = 1/2$ enters on the left, this polarization is preserved throughout its trajectory (upper dashed curve in Fig. 2), and the same is true for $S_z = -1/2$ (lower dashed curve). By rotating the analyzer by 90° about the y axis, it becomes an analyzer of the x component of spin polarization S_x.

We shall treat the analyzers as quantum mechanical devices, with the following simplified notation for the corresponding wave functions. Let Z be the wave function for the S_z analyzer before passage of the particle, and Z^+ and Z^- the functions resulting from the passage of a particle with $S_z = +1/2$ and $-1/2$, respectively. For the S_x analyzer the corresponding symbols are X, X^+ ($S_x = 1/2$) and X^- ($S_x = -1/2$). The spin states of the particle will be denoted by α and β for $S_z = +1/2$ and $-1/2$, and γ and δ for $S_x = 1/2$ and $-1/2$, respectively, with phases chosen so that

$$\begin{aligned} \alpha &= (\gamma + \delta)/\sqrt{2}, \qquad \beta = (\gamma - \delta)/\sqrt{2} \\ \gamma &= (\alpha + \beta)/\sqrt{2}, \qquad \delta = (\alpha - \beta)/\sqrt{2} \end{aligned} \tag{4.1}$$

The part of the particle wave function associated with its position in space, in contrast to its spin, will be omitted, as we make no use of it in the following discussion. The transformation in the wave function of the particle plus analyzer when the particle passes through is given by

$$\alpha Z \to \alpha Z^+, \qquad \beta Z \to \beta Z^- \tag{4.2}$$

in the case of the z analyzer, and

$$\gamma X \to \gamma X^+, \qquad \delta X \to \delta X^- \tag{4.3}$$

in the case of the x analyzer.

4.2. Some Consistent Histories and Associated Probabilities

Consider a situation, Fig. 3, in which a particle with initial polarization $S_x = 1/2$ passes through a Z analyzer followed by an X analyzer, and let t_0, t_1, and t_2 be times such that the particle is to the left of both analyzers,

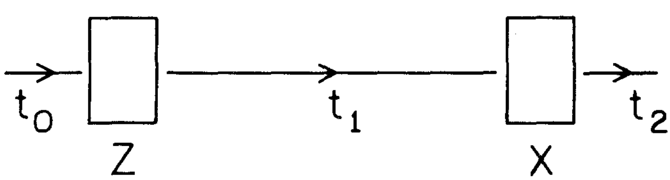

Fig. 3. Particle path through two successive spin polarization analyzers, one for S_z and one for S_x.

in between, and to the right of both, respectively. The time development to the total wave function, in analogy with (3.1) to (3.3), is

$$t = t_0: \quad \Psi_0 = \gamma Z X \tag{4.4}$$

$$t = t_1: \quad \Psi_1 = (\alpha Z^+ X + \beta Z^- X)/\sqrt{2} \tag{4.5}$$

$$t = t_2: \quad \Psi_2 = (\gamma Z^+ X^+ + \delta Z^+ X^- + \gamma Z^- X^+ - \delta Z^- X^-)/2 \tag{4.6}$$

We shall study various histories which commence at t_0 with the initial state D corresponding to Ψ_0 and terminate at t_2 with the event F that the z and x analyzers are in the states Z^+ and X^+, respectively. The intermediate events will be of the form A_j, the particle in state α, and Γ_j, the particle in state γ at the time t_j.

The consistency of various histories and the corresponding probabilities can be worked out by the same methods used in Section 3, and the details will be found in Appendix C. It turns out that both of the histories

$$D \to \mathrm{A}_1 \to F \tag{4.7}$$

$$D \to \Gamma_1 \to F \tag{4.8}$$

are consistent, and

$$P(\mathrm{A}_1 \mid D \wedge F) = 1 \tag{4.9}$$

$$P(\Gamma_1 \mid D \wedge F) = 1 \tag{4.10}$$

In words, (4.9) asserts that given the initial state and the fact that at the time t_2 the z and x analyzers are in the states Z^+ and X^+, one can be certain that at the time t_1 when the particle was between the two counters it had a polarization $S_z = 1/2$; (4.10) is the assertion that under the same conditions one can be certain that $S_x = 1/2$ at the time t_1.

The simultaneous validity of (4.9) and (4.10) comes as somewhat of a surprise, and we shall discuss this in some detail in Section 4.3 below. However, it is well to note at once that the consistent history approach does *not* allow us to combine (4.9) and (4.10) so as to deduce the consequence, obviously correct for *classical* probabilities, that

$$P(\mathrm{A}_1 \wedge \Gamma_1 \mid D \wedge F) = 1 \tag{4.11}$$

In fact the histories (4.7) and (4.8) are incompatible in the sense (Section 2.2) that events in one cannot be combined with events in the other to form a single consistent history. Indeed, they cannot even be combined into a single history given (2.3) and the remarks which follow it, since A_1 and Γ_1 do not commute. To get around this problem, we introduce a second time $t_{1.1}$ which is slightly later than t_1 but still before the particle reaches the x

analyzer. Evidently the two histories

$$D \rightarrow \mathrm{A}_{1.1} \rightarrow F \tag{4.12}$$

$$D \rightarrow \Gamma_{1.1} \rightarrow F \tag{4.13}$$

are both consistent. In addition (4.7) and (4.13) are compatible in the sense that the combined history

$$D \rightarrow \mathrm{A}_1 \rightarrow \Gamma_{1.1} \rightarrow F \tag{4.14}$$

is consistent (Appendix C), whereas

$$D \rightarrow \Gamma_1 \rightarrow \mathrm{A}_{1.1} \rightarrow F \tag{4.15}$$

is inconsistent, so that (4.8) and (4.12) are incompatible. In addition

$$P(\mathrm{A}_1 \wedge \Gamma_{1.1} \mid D \wedge F) = 1 \tag{4.16}$$

in words, given the initial and final states, it is certain that $S_z = 1/2$ was the case at one time and $S_x = 1/2$ was the case at a slightly later time, provided both times were in the interval when the particle was between the counters.

While (4.16) can be calculated directly using the definition (2.12), see Appendix C, it is worthwhile noting that it is also an immediate consequence of a standard probabilistic inference based on (4.9) and

$$P(\Gamma_{1.1} \mid D \wedge F) = 1 \tag{4.17}$$

which is the counterpart of (4.10) for the history (4.13). The reason why such reasoning is valid in this case, but that leading to (4.11) is not, is that (4.7) and (4.13) are compatible, which is the same thing as saying that both (4.9) and (4.17) refer to probabilities of events in the *same consistent family*, namely, the smallest family containing (4.14).

Finally it is amusing to note that if Θ_j is the event $S_y = 1/2$ at time t_j, then in an obvious extension of the previous notation,

$$D \rightarrow \mathrm{A}_1 \rightarrow \Theta_{1.1} \rightarrow \Gamma_{1.2} \rightarrow F \tag{4.18}$$

is a consistent history, and

$$P(\mathrm{A}_1 \wedge \Theta_{1.1} \wedge \Gamma_{1.2} | D \wedge F) = \tfrac{1}{2} \tag{4.19}$$

We leave the verification of these assertions, as well as the question of what else might be consistently inserted between A_1 and $\Gamma_{1.2}$, as exercises to the reader.

4.3. Discussion of the Results of Section 4.2

We begin with (4.9), which asserts that $S_z = 1/2$ at the time t_1 when the particle is between the two analyzers, given D and F. Many physicists would accept this result, but obtain it through some variant of the following reasoning:

A measurement takes place when the particle passes through the first analyzer, and this results in the collapse of the wave function (or its replacement by an appropriate density matrix). But since we know from F that the S_z analyzer is in the state Z^+ at time t_2, and as it cannot have changed between t_1 and t_2, the appropriate wave function at time t_1 is

$$\Psi_1^\alpha = \alpha Z^+ X \tag{4.20}$$

and thus the probability of A_1 is

$$\langle \Psi_1^\alpha | A_1 | \Psi_1^\alpha \rangle = 1 \tag{4.21}$$

in agreement with (4.9).

On the other hand, this sort of reasoning does not yield (4.10), for

$$\langle \Psi_1^\alpha | \Gamma_1 | \Psi_1^\alpha \rangle = 1/2 \tag{4.22}$$

and not 1. But then what is the correct answer (if any) for the probability that $S_x = 1/2$ at time t_1, given D and F? We shall argue that the correct answer is (4.10), and the difficulty with (4.22) is that the reasoning which led to (4.20), while valid for predicting the result (4.9), is of little use when it comes to answering the question addressed by (4.10) because it makes no use of the relevant piece of information, contained in F, that the result of a later measurement of S_x yielded the value $+1/2$ and not $-1/2$. By contrast, the consistent history approach makes important use of the information in the S_x measurement. Had the final state of the X analyzer been X^- instead of X^+, (4.10) would be 0 instead of 1.

From a physical point of view (4.10) simply reflects the idea that a measurement can indicate a property of the measured system of a time before the measurement takes place, given an appropriate construction of the measuring device and an appropriate dynamics for it and the measured system. Thus, if the S_x analyzer indicates that $S_x = 1/2$, it is reasonable to suppose that the particle had this property when it entered the analyzer, and therefore at earlier times as well, as long as it was moving in a region free of magnetic fields. In particular, with reference to Fig. 2, the reasoning employed to answer question 1 in Section 3.2 indicates that when one of the two counters C_a or C_b is triggered, an instant earlier the particle was on its way toward this counter and not the other, and therefore, given the construction of the analyzer, it already had the corresponding spin polarization. Thus if one accepts the fact that the answer to question 1 of Section 3.2 provided by consistent histories is physically reasonable, it is hard to evade the conclusion that the particle entering a polarization analyzer of the type shown in Fig. 2 had the indicated polarization before it entered. Or at the very least the physical reasons for this are just as strong as those

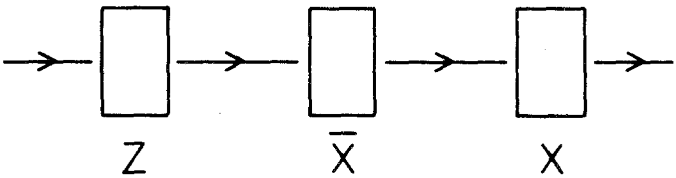

Fig. 4. A second S_x analyzer placed between the two analyzers of Fig. 3.

which lead one to conclude that the particle has this polarization after it leaves. But this puts (4.9) and (4.10) on an equivalent footing: the former is true because of an earlier measurement, the latter is true because of a later measurement.

But is it physically meaningful to talk about the value of S_x *before* it is measured? Perhaps not, but then it would seem to be just as meaningless to talk about the value of S_z *after* it is measured, and if neither is meaningful, what is meant by measurement? However, as it might help in choosing between (4.10) and (4.22), we are happy to introduce a measurement of S_x which takes place at the intermediate time t_1 by means of an analyzer $\bar{X}$ placed between the other two (Fig. 4). One can then ask: given the initial state $\gamma Z\bar{X}X$ at t_0 and the fact that the first and last analyzers are in the states Z^+ and X^+ at time t_2, what is the probability that at this same time the intermediate analyzer will be in the state $\bar{X}^+$?

As only a single time (other than t_0) is involved, the standard statistical interpretation (Section 2.3) can be used to predict the following results if the experiment is repeated many times:

(i) The result Z^+X^+ occurs in $1/4$ of the cases, just as it does when the analyzer $\bar{X}$ is absent.

(ii) Every time Z^+X^+ occurs, the intermediate analyzer is found to be in the state $\bar{X}^+$, and not $\bar{X}^-$.

What (ii) implies, taken at face value, is an experimental confirmation of (4.10). One can try and evade this by claiming that the intermediate analyzer $\bar{X}$ has "perturbed" the system and "created" something which would not have been present had the analyzer been absent. And given that quantum measurements can, indeed, perturb the system being measured, this claim must be given serious consideration.

Certainly if (i) were *not* true, i.e., if the presence of $\bar{X}$ altered the statistics of the X and Z analyzers, one would have direct evidence for such a perturbation. But (i) indicates that whatever perturbations may exist, they do not show up in the macroscopic measurement results. To go beyond this requires some sort of theoretical analysis. Fortunately the consistent history approach provides the mathematical tools needed for a rather detailed analysis of what is and what is not perturbed; see Section 5. When applied to the case at hand, and given the idealizations necessary to permit an explicit mathematical analysis, the result is that what the $\bar{X}$ analyzer measures would have been there in its absence (in an appropriate sense; see the discussion of perturbations in Section 5.4). Of course this analysis cannot rule out the possibility that a different approach to analyzing measurements, not based on consistent histories, might come to a different conclusion about the perturbation produced by $\bar{X}$.

What of the Heisenberg uncertainty relations or at least their counterpart for spin polarizations? Are not these inconsistent with the truth of (4.9) *and* (4.10) *together*? Indeed, it is not hard to show that if Ψ_1^α is *any* normalized wave function which yields (4.21), (4.22) is a necessary consequence. However, all this shows is that (4.9) and (4.10) cannot both be calculated using formulas of the type (4.21) and (4.22), at least not with the same wave function. And as already noted in Section 3.3, within the consistent history approach it is not very natural to associate a unique wave function with a physical system at every instant of time. To be sure, within the consistent history interpretation the Heisenberg uncertainty relations apply in appropriate circumstances, but these do not include the circumstance addressed by (4.9) and (4.10).

Whatever the formal merit or physical plausibility of each of the results (4.9) and (4.10) separately, many physicists will still find their simultaneous affirmation counterintuitive. While all classical analogies for spin-1/2 particles are bound to be somewhat misleading, we think the following one may, at least, place these intuitive difficulties in a new light. Suppose that the classical analog of $S_z = 1/2$ is "the z component of internal angular momentum of the particle is positive," and similarly $S_x = 1/2$ is analogous to a positive x component. Imagine that the classical particle passes successively through two measuring devices, the first of which measures the sign of the z component of the internal angular momentum without changing it, while producing unknown perturbations of the x component, while the second measures the sign of the x component without changing it, but perturbs the z component. Given that the two measurements yield "$S_z = 1/2$" and "$S_x = 1/2$" in the sense of the analogy, one can at once conclude that *both* of these results are valid at *all* times when the particle is between the two devices. From this perspective there is nothing odd about the simultaneous truth of (4.9) and (4.10), nor is (4.16) peculiar. Instead it is the refusal of the consistent history approach to countenance the plausible inference (4.11), and its rejection of (4.15) as "inconsistent" which seems to be odd!

The properties of the classical measuring devices just discussed were deliberately chosen to be in some sense analogous to those of the ideal quantum measuring devices introduced in Section 5 below. It is generally accepted that quantum measurements always produce irreducible perturbations of the system being measured. The result of the consistent history analysis of Section 5 is that one can at least imagine ideal quantum measuring devices which do *not* perturb the quantity they are designed to measure, while they do perturb other quantities. Having said this, it is well to emphasize that the analogy of the preceding paragraph must not be pressed too far; quantum physics is never really the same as classical physics, including classical physics with a stochastic component.

5. IDEALIZED MEASUREMENTS

5.1. Introduction

Real laboratory instruments which amplify the effects of atomic processes to produce signals easily perceptible by human beings are complicated objects involving vast numbers of atoms, and are operated under conditions where thermodynamic irreversibility plays an important role. A direct attack on the quantum mechanics of such an apparatus runs into all the conceptual and technical difficulties of the many-body problem and statistical irreversibility in addition to the problem of quantum interpretation. For this reason a ruthless oversimplification seems necessary in order to get to the heart of the basic quantum problem. The idealized quantum measuring apparatuses or "indicator devices" introduced below are consistent with the use of projection operators to represent physical events and with unitary time transformations, but with few other physical principles; in particular, all conservation laws (except for "conservation of probability," embodied in unitarity) are ignored. We do not suggest that such devices can actually be constructed in the laboratory, or that they would be useful if they could be. Their role is one of conceptual clarification, and for this they are well suited because their simplicity permits a detailed quantum analysis within the consistent history interpretation. One can, in particular, ask whether the measurement perturbs the system being measured, and if so in what way. One can inquire whether the system had the property indicated by the measurement at times before and at times after the measurement took place. As long as questions of this sort can be embodied in strings of events which form a consistent history, this approach to interpretation provides definite (statistical) answers which, we believe, can be the source of genuine insight into the problem of "real measurements," as well as the role of measurements in the "orthodox" approach to quantum interpretation.

5.2. A Single Ideal Measurement

Let S be a quantum system described using a Hilbert space S with a unitary time transformation $U(t',t)$. Let $[A_\tau^\alpha]$ be an event set at the time $t=\tau$, with $\alpha = 0,1,2,\ldots,N-1$ the N distinct, mutually exclusive possibilities. Let I be a quantum-mechanical "indicator" described by an N-dimensional Hilbert space I with orthonormal basis ξ^β, $\beta = 0,1,\ldots,N-1$. The combined system S_c of I and S is associated with the tensor product

$$S_c = I \otimes S \tag{5.1}$$

and we employ the usual convention that the symbol A can be used both for an operator on S and the corresponding operator $1 \otimes A$ on S_c; similarly b can denote an operator on I or the corresponding $b \otimes 1$ on S_c.

The time transformation U_c for S_c is defined in terms of U for S in the following way. Let τ' be a time slightly later than τ, with the difference so small that

$$U(\tau',\tau) = 1 \tag{5.2}$$

is an adequate approximation. Then for t and t' both less than τ or both greater than τ', define

$$U_c(t',t) = U(t',t) \tag{5.3}$$

while for

$$t \leqslant \tau < \tau' \leqslant t' \tag{5.4}$$

let

$$U_c(t',t) = U(t',\tau)U_{c\tau}U(\tau,t) \tag{5.5}$$

$$U_c(t,t') = U(t,\tau)U_{c\tau}^{-1}U(\tau,t') \tag{5.6}$$

where

$$U_{c\tau} = \sum_{\alpha=0}^{N-1} s^\alpha \otimes A_\tau^\alpha \tag{5.7}$$

$$U_{c\tau}^{-1} = \sum_{\alpha=0}^{N-1} s^{-\alpha} \otimes A_\tau^\alpha \tag{5.8}$$

with the $\{s^\alpha\}$ operators on I defined by

$$s^\alpha \xi^\beta = \xi^{\beta+\alpha} \tag{5.9}$$

the sum $\beta+\alpha$ understood as modulo N, and consistent with this, $-\alpha$ in (5.8) understood as $N-\alpha$. In principle $U(t',\tau')$ should appear in (5.5) but (5.2) justifies replacing it with $U(t',\tau)$, and a similar comment applies to (5.6).

The intuitive interpretation of U_c is that except for the short interval from τ to τ', I and S are decoupled, and I does not change with time. During this interval, which is so short that S would undergo a negligible change if left to itself, the state of I changes from ξ^{β} to $\xi^{\beta+\alpha}$ if the situation A_{τ}^{α} exists (the event A_{τ}^{α} occurs) in S at this instant. Thus if at some time before τ I is in one of the states ξ^{β}, say, ξ^{0}, its state at some later time after τ' will indicate the situation in S at time τ. This intuitive picture is both confirmed and made more precise by the detailed analysis which follows.

Given a history $\mathscr{H}$ of S in the form (2.2), with

$$t_0 < \tau < \tau' < t_f \tag{5.10}$$

and none of the t_k inside the interval from τ to τ', the *parallel history* $\mathscr{H}_c$ of S_c is defined as

$$D^c \to E_1 \to E_2 \to \cdots \to E_n \to F^c \tag{5.11}$$

where

$$D^c = d \otimes D, \qquad F^c = f \otimes F \tag{5.12}$$

and E_j in (5.11) denotes, of course, $1 \otimes E_j$. Here d and f are the initial and final states (or events) of the indicator I, and in what follows we shall always assume that

$$d = r^0, \qquad f = 1 \tag{5.13}$$

where for $\beta = 0, 1, \ldots, N-1$,

$$r^{\beta} = |\xi^{\beta}\rangle\langle\xi^{\beta}| \tag{5.14}$$

In addition it is convenient to introduce a new event set $[J_i^{\alpha}]$ at a time t_i lying between τ' and t_f, with

$$J_i^{\alpha} = r^{\alpha} \tag{5.15}$$

For notational convenience we shall always let $t_i = t_f$ though the results do not depend on this identification.

If $\mathscr{F}$ is a family of histories of S, (2.6), we shall use the symbol $\mathscr{F}_c$ for the family of parallel histories of S_c, and $\mathscr{F}_c^*$ for the larger family constructed by adding the event set $[J_i^{\alpha}]$, that is,

$$D^c \to [E_1^{\alpha}] \to \cdots [E_n^{\alpha}] \to [J_i^{\alpha}] \to F^c \tag{5.16}$$

Lest the reader be suspicious that the asymmetric choice of d and f in (5.13) together with the additional event set $[J_i^{\alpha}]$ at the end and not the beginning of the history represent a surreptitious insertion of an "arrow of time" into the interpretation, we remark that this procedure is a matter of technical convenience. The same results in a completely symmetric (but somewhat lengthier) form could be obtained by setting $d = 1$ and introducing an additional event set $[J_h^\alpha]$ at a time t_h between t_0 and τ, with $J_h^\alpha = r^\alpha$.

We are interested in the relationship of the probabilities of histories $\mathscr{H}$ of S and their parallels $\mathscr{H}_c$ of S_c, with or without an additional event J_i^α. For their study the following identity is extremely useful; its derivation is in Appendix D:

$$\begin{aligned}\mathrm{Tr}_c[\hat{J}_i^\gamma \hat{E}_n^c \ldots \hat{E}_2^c \hat{E}_1^c \hat{D}^c \hat{G}_1^c \hat{G}_2^c \ldots \hat{G}_n^c \hat{F}^c] \\ = \mathrm{Tr}[\hat{E}_n \ldots \hat{E}_{m+1} \hat{A}_\tau^\gamma \hat{E}_m \ldots \hat{E}_1 \hat{D} \hat{G}_1 \ldots \hat{G}_m \hat{A}_\tau^\gamma \hat{G}_{m+1} \ldots \hat{G}_n \hat{F}]\end{aligned} \tag{5.17}$$

where m is the index such that

$$t_m \leqslant \tau < \tau' \leqslant t_{m+1} \tag{5.18}$$

Here E_j and G_j, which may be identical, are any elements of the event set $[E_j^\alpha]$. The Heisenberg operators inside the left trace over states of S_c are defined by formulas of the type

$$\hat{E}_k^c = U_c(t_r, t_k) E_k U_c(t_k, t_r) \tag{5.19}$$

whereas the trace on the right side is over states of S, and the Heisenberg operators are defined by (2.9); note that $\hat{E}_k^c$ is in general *not* equal to $1 \otimes \hat{E}_k$. Other than this, the arguments of the two traces differ in that $\hat{J}_i^\gamma$ appears only on the left, while on the right side $\hat{A}_\tau^\gamma$ has been inserted twice at the appropriate point in the time order. In applications of (5.17) there will typically be an additional operator $\hat{J}_i^\alpha$ between $\hat{G}_n^c$ and $\hat{F}^c$ on the left side. However, since $\hat{J}_i^\alpha$ commutes with $\hat{F}^c$ [see (5.13)] it is evident that this trace will vanish for $\alpha \neq \gamma$, and be identical to (5.17) for $\alpha = \gamma$.

Theorem 1. Let $\mathscr{F}$ be a consistent family of histories of S compatible with the event set $[A_\tau^\alpha]$. Then the family $\mathscr{F}_c$ for S_c is consistent and compatible with $[A_\tau^\alpha]$, and so is $\mathscr{F}_c^*$.

As $\mathscr{F}_c$ is a subset of $\mathscr{F}_c^*$, we need only study the latter. The proof of consistency requires checking the counterpart of (2.19) for each choice of k, including the "extra" event $k = i$. But this last is trivial since $\hat{J}_i^\gamma$ commutes with $\hat{F}^c$. If k takes any other value, (5.17) shows that the consistency condition for $\mathscr{F}_c^*$ can be reexpressed as a consistency condition for $\mathscr{F}$ with the presence of the additional event $\hat{A}_\tau^\gamma$ at time τ. But as $\mathscr{F}$ is compatible with $[A_\tau^\alpha]$, the real part of the trace on the right side of (5.17) vanishes, and thus also that on the left. This takes care of the cases in which the event at t_i is one of the J_i^γ. If it is the sum of two or more, the corresponding trace is the sum of traces corresponding to the separate J_i^γ, so the real part will again vanish. The proof that $\mathscr{F}_c^*$ is compatible with $[A_\tau^\alpha]$ comes from noting that if $\mathscr{F}$ does not already include the event set $[A_\tau^\alpha]$, it can be enlarged to a consistent family which does include $[A_\tau^\alpha]$, and the above argument shows the consistency of the enlarged $\mathscr{F}_c^*$ corresponding to this enlarged $\mathscr{F}$.

Theorem 2. Let $\mathscr{H}$ be a consistent history of S of the form (2.2) which is compatible with the event set $[A_\tau^\alpha]$, and $\mathscr{H}_c$ its parallel, (5.11), for S_c. Then it is the case that

$$P(\mathscr{H}_c \mid D^c \wedge F^c) = P(\mathscr{H} \mid D \wedge F) \tag{5.20}$$

that is to say,

$$P(E_1 \wedge \cdots E_n \mid D^c \wedge F^c) = P(E_1 \wedge \cdots E_n \mid D \wedge F) \tag{5.21}$$

As pointed out in Section 2.2, (5.20) is simply an alternative way of writing (5.21). To establish this result, note that $\mathscr{H}_c$ is consistent, by Theorem 1, and that $w(\mathscr{H}_c)$ [see (2.10)] is the sum of (5.17) over the N values of γ in the case where $G_j = E_j$ for all j. As $\mathscr{H}$ is compatible with $[A_\tau^\alpha]$, summing the right side of (5.17) over γ yields the trace of the corresponding expression with the two $\hat{A}_\tau^\gamma$ replaced by 1, i.e., use the counterpart of (2.14). Thus $w(\mathscr{H}_c)$ is the same as $w(\mathscr{H})$, and for the same reason $w(D^c \wedge F^c)$ is the same as $w(D \wedge F)$.

Theorem 3. The family of histories

$$D^c \to [A_\tau^\alpha] \to [J_i^\alpha] \to F^c \tag{5.22}$$

of S_c is consistent (whether or not the corresponding $D \to [A_\tau^\alpha] \to F$ is a consistent family for S), and

$$P(A_\tau^\alpha \wedge J_i^\gamma \mid D^c \wedge F^c) = 0, \qquad \text{for} \quad \alpha \neq \gamma \tag{5.23}$$

$$P(A_\tau^\alpha \wedge J_i^\alpha \mid D^c \wedge F^c) = P(A_\tau^\alpha \mid D^c \wedge F^c) = P(J_i^\alpha \mid D^c \wedge F^c) \tag{5.24}$$

consequently

$$P(J_i^\gamma \mid D^c \wedge A_\tau^\alpha \wedge F^c) = \delta_{\alpha\gamma} = P(A_\tau^\alpha \mid D^c \wedge J_i^\gamma \wedge F^c) \tag{5.25}$$

As usual these equations are understood as applicable only when the probabilities of the events of the right of the vertical bar are nonzero; in particular, (5.23) and (5.24) require that $\mathrm{Tr}_c[\hat{D}^c \hat{F}^c]$ not vanish, the left equality in (5.25) assumes that $P(A_\tau^\alpha \mid D^c \wedge F^c)$ is nonzero, etc. The proof that (5.22) is consistent employs (5.17) in the same manner as the previous proofs, except that now $n = 1$, and E_1, or G_1, belongs to $[A_\tau^\alpha]$, and the derivation of (5.23) and (5.24) follows the same route. We omit the details. The results in (5.25) are consequences of (5.23), (5.24), and the definition of conditional probabilities.

In words, what (5.25) asserts is that if the event A_τ^α occurs ("exists") at time τ in S_c, we can be sure that at the later time t_i the indicator will be in the state ξ^α and not in any other ξ^γ with $\gamma \neq \alpha$. Conversely, if the indicator is in the state ξ^γ at the later time, A_τ^γ occurred at the earlier time, and A_τ^α for $\alpha \neq \gamma$ did not occur. A particular application of this result is the following:

Theorem 4. If $\mathscr{H}_c$ is a consistent history of S_c of the type (5.11) which is compatible with the event set $[A_\tau^\alpha]$, then

$$P(J_i^\gamma \mid \mathscr{H}_c \wedge A_\tau^\alpha) = \delta_{\alpha\gamma} = P(A_\tau^\alpha \mid \mathscr{H}_c \wedge J_i^\gamma) \tag{5.26}$$

The first step of the proof consists in showing with the help of (5.17) that $\mathscr{H}_c$ is compatible with $[A_\tau^\alpha]$ and $[J_i^\gamma]$ together; we omit the details. Hence $\mathscr{H}_c \wedge A_\tau^\alpha \wedge J_i^\alpha$ belongs to a consistent family which contains (5.22) as a subset, and (5.26) follows from (5.25) by an argument applicable to classical probabilities: from $P(A \mid B)$ equal to 1 or 0 we can infer that $P(A \mid B \wedge C)$ equals 1 or 0, respectively.

In words, (5.26) asserts that if A_τ^α occurs along with certain other events, and together they form a consistent history of S_c, then at the time t_i the indicator will certainly be in the state ξ^α. Conversely, if the indicator is in state ξ^α at t_i, one can be sure that A_τ^α occurred, provided the consistency condition is satisfied. Note that Theorem 1 implies that the consistency of the parallel history $\mathscr{H}$ and its compatibility with $[A_\tau^\alpha]$ in the system S is sufficient (though it is not necessary) to establish the condition required in Theorem 4.

5.3. Several Ideal Measurements

The approach and the results of Section 5.2 can be generalized to the case of several ideal measurements on a system S taking place at the successive times $\tau_1, \tau_2, \ldots, \tau_\nu$, with

$$t_0 < \tau_1 < \tau_2 < \cdots < \tau_\nu < t_f \tag{5.27}$$

Let $[A_\mu^\alpha]$ be an event set at the time τ_μ, with $\alpha = 0, 1, \ldots, N_\mu - 1$. Then the μth indicator device I_μ is described using an N_μ-dimensional Hilbert space I_μ with an orthonormal basis ξ_μ^β, $0 \leqslant \beta \leqslant N_\mu - 1$. The combined system S_c of S plus the ν indicators is associated with the tensor product space

$$S_c = I \otimes S = I_1 \otimes I_2 \otimes \cdots I_\nu \otimes S \tag{5.28}$$

where I is the product space for the set of indicators.

The time transformation U_c of S_c is related to U for S as follows. For each τ_μ let τ'_μ be a time which is slightly later, with the interval so short that

$$U(\tau_\mu, \tau'_\mu) = 1 \tag{5.29}$$

is an adequate approximation (and of course τ'_μ is earlier than $\tau_{\mu+1}$). We shall assume that all the events in the histories we wish to study fall outside these short intervals. For both t and t' between τ'_μ and $\tau_{\mu+1}$ for some μ, or both less than τ_0 or greater than τ'_ν, define

$$U_c(t',t) = U(t',t) \tag{5.30}$$

(where the symbol on the right stands for $1 \otimes 1 \otimes \cdots U(t',t)$, and a similar convention applies to other operators). Next, for t and t' satisfying

$$\tau'_{\mu-1} \leqslant t \leqslant \tau_\mu < \tau'_\mu \leqslant t' \leqslant \tau_{\mu+1} \tag{5.31}$$

let

$$U_c(t',t) = U(t',\tau_\mu)U_{c\mu}U(\tau_\mu,t) \tag{5.32}$$

$$U_c(t,t') = U(t,\tau_\mu)U_{c\mu}^{-1}U(\tau_\mu,t') \tag{5.33}$$

where

$$U_{c\mu} = \sum_{\alpha=0}^{N_\mu-1} s_\mu^\alpha \otimes A_\mu^\alpha \tag{5.34}$$

$$U_{c\mu}^{-1} = \sum_{\alpha=0}^{N_\mu-1} s_\mu^{-\alpha} \otimes A_\mu^\alpha \tag{5.35}$$

with $\{s_\mu^\alpha\}$ operators on I_μ defined by

$$s_\mu^\alpha \xi_\mu^\beta = \xi_\mu^{\beta+\alpha} \tag{5.36}$$

and the sum $\alpha + \beta$ understood modulo N_μ; similarly $-\alpha$ in (5.35) is understood as $N_\mu - \alpha$. For other choices of t and t' (always outside the intervals from τ_μ to τ'_μ), U_c can be obtained using the counterpart of (2.7). The intuitive interpretation of these equations is the same as in Section 5.2, except that S now interacts with several different indicators at successive times. The property "measured" by the μth indicator is, of course, determined by the event set $[A_\mu^\alpha]$.

Consider a history $\mathscr{H}$ of S of the form (2.2) with initial and final times satisfying (5.27). Its parallel $\mathscr{H}_c$ for S_c is (5.11), with d and f in (5.12) given by

$$d = r_1^0 \otimes r_2^0 \otimes \cdots r_\nu^0, \qquad f = 1 \tag{5.37}$$

where

$$r_\mu^\alpha = |\xi_\mu^\alpha\rangle\langle\xi_\mu^\alpha| \tag{5.38}$$

As in Section 5.2, it is convenient to introduce an event set $[J_i^a]$ at some time t_i between τ'_ν and t_f, where a stands for

$$a = \alpha_1, \alpha_2, \ldots, \alpha_\nu \tag{5.39}$$

and

$$J_i^a = r_1^{\alpha_1} \otimes r_2^{\alpha_2} \otimes \cdots r_\nu^{\alpha_\nu} \tag{5.40}$$

is an operator on I; of course the same symbol denotes $J_i^a \otimes 1$ on S_c. Again we assume that $t_i = t_f$ as a matter of notational convenience, but the results do not depend on this identification.

Given a family $\mathscr{F}$ of histories of S of the form (2.6), $\mathscr{F}_c$ will denote the corresponding family (5.11) of parallel histories of S_c, and $\mathscr{F}_c^*$ the latter argumented with the additional event set $[J_i^a]$:

$$D^c \to [E_1^\alpha] \to [E_2^\alpha] \to \cdots [E_n^\alpha] \to [J_i^a] \to F^c \tag{5.41}$$

It is worth emphasizing that the event sets of type $[E_k^\alpha]$ play a very different role from the $[A_\mu^\alpha]$. The latter enter explicitly into the *dynamics* of S_c as part of the definition of U_c, whereas the former do not. Altering the $[E_k^\alpha]$ changes the set of physical *questions* addressed by the theory without altering the dynamics of either S or S_c, while changing the $[A_\mu^\alpha]$ alters the dynamics of S_c (though of course not that of S). Naturally, nothing prevents setting some (or all) of the $[E_k^\alpha]$ equal to some of $[A_\mu^\alpha]$ when the occurrence of the latter is among the questions of interest.

The counterpart of the trace formula (5.17) is now (see Appendix D)

$$\begin{aligned} \mathrm{Tr}_c[\hat{J}_i^a \hat{E}_n^c \ldots \hat{E}_2^c \hat{E}_1^c \hat{D}^c \hat{G}_1^c \hat{G}_2^c \ldots \hat{G}_n^c \hat{F}^c] &= \mathrm{Tr}[\ldots \hat{A}_\nu^{\alpha_\nu} \ldots \hat{E}_k \ldots \hat{A}_1^{\alpha_1} \ldots \hat{D} \ldots \\ &\quad \times \hat{A}_1^{\alpha_1} \ldots \hat{G}_k \ldots \hat{A}_\nu^{\alpha_\nu} \ldots \hat{F}] \end{aligned} \tag{5.42}$$

where on the right side all the operators $\hat{E}_n \ldots \hat{E}_1$ occur to the left of $\hat{D}$, with the $\hat{A}_\mu^{\alpha_\mu}$ inserted *in the correct time order* [see (5.17) and (5.18) for a particular case] while the operators $\hat{G}_1 \ldots \hat{G}_n$ occur to the right of $\hat{D}$, again with the $\hat{A}_\mu^{\alpha_\mu}$ inserted in the correct time order. The values of $\alpha_1, \alpha_2, \ldots$ on the right side of (5.42) correspond to a on the left; see (5.39).

Theorem 5. Let $\mathscr{F}$ be a consistent family of histories of S compatible with all the event sets $[A_\mu^\alpha]$, $1 \leqslant \mu \leqslant \nu$, together. Then the family $\mathscr{F}_c$ for S_c is consistent and compatible with all the $[A_\mu^\alpha]$ together, and so is $\mathscr{F}_c^*$.

The proof makes use of (5.42) in a manner parallel to the use of (5.17) in the proof of Theorem 1, so we omit the details.

Theorem 6. Let $\mathscr{H}$ be a consistent history of S of the form (2.2) which is compatible with all the event sets $[A_\mu^\alpha]$, $1 \leqslant \mu \leqslant \nu$, together, and

$\mathscr{H}_c$ its parallel, (5.11), for S_c. Then their probabilities are related by (5.20) or, equivalently, (5.21).

The proof is parallel to that of Theorem 2, and makes use of the fact that

$$\sum_a J_i^a = 1 \tag{5.43}$$

where $\sum_a$ stands for the multiple sum over $\alpha_1, \alpha_2, \ldots, \alpha_\nu$.

The notation

$$\mathscr{A}^b = A_1^{\beta_1} \wedge A_2^{\beta_2} \wedge \cdots A_\nu^{\beta_\nu} \tag{5.44}$$

where b stands for $\beta_1, \beta_2, \ldots, \beta_\nu$, and

$$\delta_{ab} = \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} \cdots \delta_{\alpha_\nu\beta_\nu} \tag{5.45}$$

is needed for the generalization of Theorem 3:

Theorem 7. The family of histories

$$D^c \to [A_1^\alpha] \to [A_2^\alpha] \to \cdots \to [A_\nu^\alpha] \to [J_i^\alpha] \to F^c \tag{5.46}$$

of S_c is consistent, and

$$P(\mathscr{A}^b \wedge J_i^a \mid D^c \wedge F^c) = 0 \qquad \text{if} \quad \delta_{ab} = 0 \tag{5.47}$$

while

$$\begin{aligned} &P(\mathscr{A}^a \wedge J_i^a \mid D^c \wedge F^c) \\ &\quad = P(\mathscr{A}^a \mid D^c \wedge F^c) = P(J_i^a \mid D^c \wedge F^c) \\ &\quad = \Lambda(a)/\textstyle\sum_a \Lambda(a) \end{aligned} \tag{5.48}$$

where

$$\begin{aligned} \Lambda(a) &= w(D \wedge A_1^{\alpha_1} \wedge A_2^{\alpha_2} \wedge \cdots A_\nu^{\alpha_\nu} \wedge F) \\ &= \mathrm{Tr}[\hat{A}_\nu^{\alpha_\nu} \ldots \hat{A}_1^{\alpha_1} \hat{D} \hat{A}_1^{\alpha_1} \ldots \hat{A}_\nu^{\alpha_\nu} \hat{F}] \end{aligned} \tag{5.49}$$

A consequence of (5.47) and (5.48) is

$$P(J_i^a \mid D^c \wedge \mathscr{A}^b \wedge F^c) = \delta_{ab} = P(\mathscr{A}^b \mid D^c \wedge J_i^a \wedge F^c) \tag{5.50}$$

It is worth noting that in the case $F = 1$, the right side of (5.48) has the somewhat simpler form:

$$P(J_i^a \mid D^c) = \mathrm{Tr}[\hat{A}_\nu^{\alpha_\nu} \ldots \hat{A}_1^{\alpha_1} \hat{D} \hat{A}_1^{\alpha_1} \ldots \hat{A}_\nu^{\alpha_\nu}]/\mathrm{Tr}[\hat{D}] \tag{5.51}$$

The proof of Theorem 7, including the consistency check and the verification of (5.47) and (5.48), is carried out using the trace formula (5.42), and once again we omit the details.

Theorem 8. If $\mathscr{H}_c$ is a consistent history of S_c of the type (5.11) which is compatible with all the event sets $[A_\mu^\alpha]$, $1 \leqslant \mu \leqslant \nu$ together, then

$$P(J_i^a \mid \mathscr{H}_c \wedge \mathscr{A}^b) = \delta_{ab} = P(\mathscr{A}^b \mid \mathscr{H}_c \wedge J_i^a) \tag{5.52}$$

The proof is parallel to that of Theorem 4, as is the intuitive significance of the result.

5.4. Discussion of the Results of Sections 5.2 and 5.3

An elementary but important observation is that the elementary indicator I of Section 5.2 measures what it was designed to measure: its state after the interaction (that is to say, the change in its state from before the interaction to afterwards) is correlated in an appropriate manner with the existence of the property of interest in the system S (as part of S_c) at the time of interest; see (5.25). Furthermore, this is not simply a matter of definition, it comes from mathematical analysis. As measurements are *not* among the fundamental interpretive elements of the consistent history approach, this result is nontrivial, and it is particularly interesting in light of the controversies and conceptual confusion which surround many discussions of quantum measurements.

Not only does S possess the measured property at the instant of interaction, but the same is true shortly before and shortly after the interaction, as can be shown by an extension of the analysis employed for Theorems 3 and 7. Here "shortly" means after the previous measurements and before the next, and that a negligible change occurs in the Heisenberg operator in S corresponding to the projection A_τ^α of interest (which is automatically true when A_τ^α is a constant of the motion). Thus the time asymmetry of the von Neumann interpretation (Chapter 3, Section 3 of his book[(1)]) is absent in the consistent history approach, with consequences which are evident in the examples of Sections 3 and 4.

The elementary nature of the indicators and the extremely simple form for their interaction with S suggest that they may well be the most "benign" form of measurement possible within a quantum context, and hence well suited for discussing the truly "irreducible" perturbations produced in a measured system by its interaction with a quantum measuring device. The consistent history approach allows one to analyze these perturbations in some detail, with the result (not very surprising) that the perturbation depends very much on what aspect of S, i.e., which consistent history, is under study.

At this point it is worth noting that the "perturbation" produced by coupling I to S depends not only on the form of this coupling, as expressed in U_c, but also on the form of the initial state d of I (and final state f, if not equal to 1). While the particular form employed in (5.13), or its counterpart (5.37), is not essential, *some* restriction is needed in order to obtain the basic consistency results of Theorems 1 and 5, which underlie Theorems 2 and 6, and the discussion below.

The simplest situation is a consistent history $\mathscr{H}$ of S which is compatible with the events being measured. Here the result (5.20) or (5.21), for either a single measurement (Theorem 2) or several (Theorem 6), can be interpreted as saying that coupling I to S produces *no perturbation whatsoever*: the probability for $\mathscr{H}$ within the isolated system S is precisely the same as that of its counterpart $\mathscr{H}_c$ in the combined system S_c. To counter the argument that identical statistics is no proof of the absence of a perturbation in a single experiment, we note that consistent histories provide a probabilistic, not a deterministic, approach to quantum interpretation, so that within this interpretation identical statistics is as close as one can come to asserting that there is "no perturbation." Any other definition would lead to the odd conclusion that S is "perturbed" even when it is isolated and does not interact with I!

In the case considered in Section 5.2, a single measurement at one time, there is always some consistent history of S (though not for an arbitrary choice of D and F) which is compatible with the event set $[A_\tau^\alpha]$, and is thus unperturbed when S is coupled to I. However, with multiple measurements at successive times it is easy to imagine situations (e.g., let S be a spin-1/2 particle subject to several measurements of S_x and S_z which alternate in time) in which there is no consistent history of S compatible with all of the $[A_\mu^\alpha]$ together. In this case every aspect of S is perturbed by the sequence of measurements.

It is of interest to note that any consistent history $\mathscr{H}$, and more generally any consistent family $\mathscr{F}$ of the form (2.6) of histories of S is "measurable" in the sense that one can choose $\nu = n$, let $\tau_k = t_k$ for each k, and let $[A_k^\alpha]$ be the same event set as $[E_k^\alpha]$. The compatibility of $\mathscr{F}$ with all the measurement event sets together is thus trivial, and hence these measurements do not perturb the histories in $\mathscr{F}$. Furthermore the occurrence or nonoccurrence of every event in one of these histories is faithfully "recorded" by the corresponding measuring device; see (5.47) and (5.48).

When perturbations do occur, there are several possibilities:

1. The history $\mathscr{H}$ of S is consistent and so is the parallel history $\mathscr{H}_c$ of S_c, but the probability of $\mathscr{H}_c$ differs from that of $\mathscr{H}$. This case seems intuitively similar to what would happen in a classical system if measuring devices produced random perturbations (as in the analogy used in Section 4.3).

2. The history $\mathscr{H}$ is consistent, but its parallel $\mathscr{H}_c$ is not. This type of perturbation seems intuitively quite different from 1, since rather than

asserting that the motion of S has changed due to its interaction with I, it implies that a question about this motion which made good sense in the absence of coupling to I is "meaningless" when this interaction is present (in the sense that the consistent history approach cannot give it any meaning). It is hard to think of a suitable classical analogy.

3. The history $\mathscr{H}$ is inconsistent, whereas its parallel $\mathscr{H}_c$ is consistent. Examples of this situation, the "inverse" of 2, are easily produced using Theorems 3 and 7, which assert that a history consisting entirely of *measured* events of S (apart from D and F) will always be consistent in S_c, even though it might be inconsistent in the isolated system S. The intuitive explanation is that the correlations between S and I produced by their interactions prevent the quantum interference effects which cause inconsistency in S by itself from affecting the corresponding sequence of events in S_c. Note that one can augment consistent histories of S_c produced in this way by adding appropriate additional events between the measured ones; (4.18) is an example.

In summary, the consistent history approach confirms the conventional wisdom that quantum measurements produce irreducible perturbations, but it also provides a detailed (though perhaps not altogether satisfying) analysis of *which* aspects of the measured system are perturbed, and in what way.

Can the analysis of Sections 5.2 and 5.3 be extended in the direction of the more realistic sorts of measurements mentioned in Section 5.1? One approach is to try and relax successively some of the drastic simplifications employed in the ideal case. The easiest to get rid of is the extremely short time interval between τ and τ'. As long as the unitary transformation carrying S_c from one side of this interval to the other is specified, there is no limitation on the length of the interval, and the consistent history approach can be used to analyze what happens at earlier or later times—or even within the interval itself, if the corresponding U_c is known. A more complicated dynamical interaction producing additional changes in S can also be tolerated, though this will complicate the relationship between histories of S and their counterparts for S_c. An indicator I with nontrivial dynamics at times lying outside the interval from τ to τ' would seem to pose no problems in principle, though the fact that the choice of d (and f) is of some importance even in the simplest case must be kept in mind.

All of this is still very far from a realistic description of even the simplest macroscopic measuring device exhibiting thermodynamic irreversibility. There is no formal difficulty in including some of the necessary tools in a consistent history approach. Thus a density matrix can be used instead of a projection operator for d, and the J_i^α could very well be projections referring to macroscopic events, as these are treated on the same footing as microscopic events in the consistent history analysis. Whether such calculations will confirm the picture suggested by Sections 5.2 and 5.3 or uncover essential flaws cannot be confidently answered in advance. There exist already, however, some arguments about the behavior of large quantum systems which can be re-interpreted from the consistent history perspective (Section 7.2.1), and seem compatible with the conclusions drawn in Sections 5.2 and 5.3.

6. CONCLUSIONS

6.1. Summary of the Consistent History Approach

Consistent histories provide a probabilistic interpretation of nonrelativistic quantum mechanics for a finite closed (i.e., isolated) physical system. The objects studied are sequences of events occurring at a succession of times, denoted by "histories." Each event is represented by an orthogonal projection operator on the Hilbert space used to describe the system, and time development corresponds to unitary transformations of this space—these, of course, are standard tools of standard quantum mechanics.

Probabilities are assigned to a history, or more generally a family of histories, provided a certain *consistency condition* (described in detail in Section 2) is satisfied. The use of this condition to make a selection among all possible event sequences is the main innovation in the consistent history approach, the aspect which separates it from other approaches to quantum interpretation, some of which are discussed below in Section 7. If consistency is satisfied, a conditional probability for intermediate events, given the initial (D) and final (F) events, is assigned through Equation (2.12). The physical interpretation is then based on these probabilities and others obtained from them by well-defined procedures. The structure also allows for probabilities conditional on the initial event alone (by setting $F = 1$), or on the final event alone ($D = 1$), or even unconditional probabilities ($F = 1 = D$), though it is not clear that the last are useful in physical applications. The whole procedure for determining consistency and assigning probabilities is explicitly independent of the sense of time: a history read "backwards" from the final event to the initial event is treated in precisely the same way as one read "forwards" from the initial event to the final.

The consistency condition is a mathematical requirement which constrains the choice of a sequence of events to which a probability can be assigned. It makes no reference at all to whether the events are microscopic or macroscopic, nor to any process of measurement, nor to whether the events are somehow "observable." (Of course, all events must be represented by orthogonal projection operators.) When consistency is satisfied it has as a consequence—indeed, this can be regarded as the definition of consistency—the fact that the probabilities assigned to histories belonging to a consistent family can be treated as if they were classical probabilities and manipulated accordingly; for example, the standard formulas for conditional probabilities are applicable. In the consistent history approach these probabilities are then given their usual classical interpretation. Thus, for example, from $P(A \mid B) = 1$ and the occurrence (or existence) of the set of events B, one can infer the occurrence (existence) of the event or set of events A. The probabilities so interpreted make physical sense, or at least this seems to be the case for the specific examples considered in Sections 3 and 4. On the other hand, *inconsistent* histories (or families of histories), examples of which are easily constructed, remain uninterpreted; they are "meaningless" in the sense that the consistent history interpretation assigns them no meaning. (For histories which are almost consistent, see the additional remarks in Section 6.2 below.)

The well-known conceptual difficulties which arise in standard quantum mechanics when the operators associated with two observables fail to commute have their counterpart in the fact that one cannot in general combine all of the events belonging to two consistent histories, or simply add additional events to a single history, and expect the result to be consistent. Even two histories with the same initial and final event may be "incompatible" in the sense just mentioned. We take the attitude (see the example in Section 4) that this incompatibility is no ground for rejecting the (probabilistic) conclusions obtained from the two histories separately, but it does rule out the possibility of combining such conclusions by the usual classical rules in order to derive certain additional results.

While measurements are not fundamental to the interpretation scheme of consistent histories (and also *because* of this fact), the latter can be used to analyze idealized, and in principle more realistic measurement processes in considerable detail, by the device of considering the measuring apparatus and the system measured as part of a single closed quantum system. One can sensibly formulate such questions as: Does the measuring instrument perturb the measured system, and if so how? Is an indication on a measuring device at some later time correctly correlated with the state of the system measured at the moment the measurement took place? Did the system have the specified property before the measurement took place, and/or afterwards? Sometimes the consistent history approach will refuse to answer the question on the grounds that it involves the discussion of events which do not form a consistent history, but in a surprisingly large number of cases, at least for the highly idealized measurements discussed in Section 5, answers are available which make good physical sense, and also throw a little light on the role of measurements in "orthodox" quantum interpretation (Section 7.1). Of particular interest is the fact that, in marked contrast to von Neumann's approach to measurement (Chapter 3, Section 3 of Ref. 1), the consistent history formulation allows one to draw inferences from the measured result about the state of the measured system before as well as after the measurement with equal ease.

6.2. Extensions and Applications

A peculiar feature of the consistent history approach is that it begins with a nonnegative weight function (2.10) which cannot be consistently interpreted in general as a (nonnormalized) classical probability, and then selects certain situations in which the classical rules apply by the criterion that these rules do in fact work. What is remarkable is not that the resulting numbers behave *formally* like probabilities, which is necessarily the case, but that they can be given a sensible physical interpretation. Now while (2.10) is a natural and fairly simple generalization of the corresponding classical expression (for events in a classical phase space with time development given by Hamilton's equations) to the noncommutative quantum case, it is hardly unique, and one wonders if there are other expressions related to temporal sequences of events, or perhaps some very different quantum structure, which can be given a physically reasonable or at least interesting probabilistic interpretation by using a similar selection process.

Another direction in which one might hope to extend the consistent history approach is to find some physical interpretation of the weights given by (2.10) when the events in question do *not* form a consistent history. One such interpretation is already implicit in Theorem 7 of Section 5: aside from normalization, the weights for histories belonging to a particular (inconsistent) family are the probabilities that the corresponding *consistent* histories *would* occur in a combined system which includes idealized measuring instruments which detect the different events in the original system at the appropriate times. However, this interpretation is neither simple nor a source of much intuition, given all the peculiarities associated with quantum measurements. Can one do better?

For reasons indicated in Section 2.2 it would be useful to have a notion of "almost consistent," or "consistent for all practical purposes." A first approximation to what is needed can be constructed quite readily: violations of (2.19) should be so small that physical interpretations based on the weights (2.10) remain essentially unchanged if the latter are shifted by amounts comparable to the former. Of course this leaves things as a matter of judgment, and the errors which might be of concern in some studies could be considered unimportant in others. Perhaps this is the best that can be done, but a more systematic approach could be of value. Obviously, this problem is not entirely unrelated to that mentioned in the previous paragraph.

There are several possible applications of the consistent history approach in its present form which could be useful in providing a better intuitive understanding of quantum mechanics. Those which immediately come to mind are the standard quantum paradoxes represented by simple gedanken experiments comparable to those considered in Sections 3 and 4: diffraction by a double slit, various forms of the Einstein–Podolsky–Rosen[2] paradox, etc. A great virtue of the consistent history approach is that as long as these problems can be "modeled" in a form simple enough so that the appropriate time transformations can be worked out (by standard quantum mechanics), the consistent history answers—including, of course, the refusal to give an answer because of inconsistency—come out by strict mathematical reasoning without appeal to physical or philosophical intuition. Whether these answers are palatable, or yield some physical insight, is of course another matter which can only be settled by considering the explicit results.

Another task to which consistent histories might make a positive contribution is that of understanding why, assuming the fundamental processes of nature are quantum mechanical, the events of everyday life (including the meter readings in physics laboratories) seem to be described so well using classical concepts. Since the consistent history approach uses the same formal machinery for macroevents as for microevents, and does not need an external observer or a *deus ex machina* to collapse wave functions, it would seem to be a good starting point for such a study. "But why the consistency condition, since the events of interest in the macroscopic domain can be represented by operators which *commute* for all practical purposes?" is an obvious question. While it is at least plausible (see von Neumann's arguments in Chapter 5, Section 4 of Ref. 1) that a set of commuting operators will suffice for macroevents *at a single time*, the appropriate generalization for a *succession* of macroscopic events in a closed system at *different* times, given that one of course wants to study the statistical correlations between them, is not obvious. The use of the corresponding Heisenberg operators referred to a common reference time does not appear promising (i.e., they are unlikely to commute; see the comments at the end of Section 7.2.2). The consistency condition may very well provide a natural and useful way of approaching such a problem: i.e., one would hope to show that large classes of histories involving appropriate macroscopic events are consistent and compatible with each other. However, to say this is to name a research program, and the actual utility of consistent histories for this purpose remains to be shown.

7. THE LITERATURE

The literature on quantum interpretation is immense, and Jammer has devoted an entire book[3] to a discussion of various points of view. Our intention here is simply to discuss certain proposals which seem particularly important or particularly relevant to the consistent history approach, and for each of these we mention only some of the published material.

7.1. Quantum Orthodoxy

While its representatives disagree among themselves at many points, there is nonetheless an easily identified mainstream of quantum interpretation, called the "orthodox view" by Wigner in a paper[4] which gives an excellent summary. A lengthier exposition of similar ideas will be found in the book by London and Bauer.[5] Evidence for its dominant place in modern physics comes from the fact that this interpretation, or at least important elements of its viewpoint, underlies the expositions found in the well-known textbooks of Landau and Lifshitz,[6] Messiah,[7] and Schiff.[8]

The principal features of quantum orthodoxy, following Wigner's exposition, are as follows:

(i) The state of a closed physical system S is completely specified by a wave function ψ which evolves in time according to the Schrödinger equation.

(ii) The wave function can be used to predict the probability of the result of a measurement carried out on S by letting it interact with an external apparatus (during which time S is no longer a closed system).

(iii) As a result of this interaction the wave function after the measurement has a form which is determined by the result of the measurement (or by an appropriate density matrix if the result of the measurement is not specified). For a suitably idealized "instantaneous" measurement this form is specified by von Neumann's "projection postulate" (see Chapter 5, Section 1 of Ref. 1), and the process by which the wave function before the measurement is changed into the wave function (or density matrix) after the measurement is often referred to as "collapsing" the former.

The above principles do not specify what is happening inside a closed system during the time intervals between measurements, and here there are some differences of opinion. One extreme is the agnostic one: quantum mechanics only predicts the results of measurements and says nothing about the system being measured. While Wigner is probably sympathetic to this point of view, von Neumann[1] appears to take a more realistic position that ψ can be used to calculate probabilities at any time, in the sense of what we have called the "standard statistical interpretation" in Section 2.3. If, however, the system is subjected to a measurement, the result will be to perturb the system so that after a measurement (of a suitably ideal form) ψ is an appropriate eigenstate of the operator corresponding to the quantity measured.

The differences just mentioned do not affect predictions about the outcome of a measurement or a set of successive measurements. Using (i), (ii), and (iii) one can work out the probability that a set of (appropriately ideal) measurements carried out at a succession of times $t_1 < t_2 < \cdots$ will yield the results $\alpha_1, \alpha_2, \ldots$; the result [see, e.g., Eq. (12) of Ref. 4] is

$$P(\alpha_1, \alpha_2, \ldots, \alpha_\nu) = \frac{\mathrm{Tr}[\hat{A}_\nu^{\alpha_\nu} \ldots \hat{A}_1^{\alpha_1}\hat{D}\hat{A}_1^{\alpha_1} \ldots \hat{A}_\nu^{\alpha_\nu}]}{[\mathrm{Tr}\,\hat{D}]} \tag{7.1}$$

where we have written the right-hand side in the notation of Section 5 above. A time-symmetric version of (7.1) has been worked out by Aharonov *et al.*[9]

Many physicists who are sympathetic with the orthodox interpretation are nonetheless dissatisfied with its reference to external measurements. Why not treat S and the measuring apparatus I as a single closed system $I + S$? Solving the Schrödinger equation for such a system will in general lead to a grotesque wave function [in our terminology; (3.3) is an example] and the problem of interpreting what this means. One approach is to employ (ii), but then one must let $I + S$ interact with an external measuring device, and one is back to the original problem. Attempts to resolve the resulting "quantum measurement problem" through reference to conscious observers or genuinely classical pieces of apparatus are not very palatable to most physicists. Some other possibilities are described in Section 7.2 below.

By contrast, the consistent history approach ascribes probabilities to sequences of events in closed systems, denies that the physical states of such systems are appropriately described by wave functions in the manner used in the orthodox approach (see Section 3.3), and does not give to measurements any fundamental role. Thus at first sight it might seem that it has nothing whatsoever in common with the orthodox interpretation. But that is misleading, for when measurements of a suitably idealized sort (in particular involving the "short time" approximation of von Neumann; see Chapter V, Section 1 of Ref. 1) are analyzed in the consistent history formalism, the measurement statistics are identical to those of the orthodox approach: (5.51) corresponds to (7.1), and (5.48) to the symmetrized version of (7.1) considered by Aharonov *et al.*[9] How can this surprising unanimity be understood, given the differences in the two approaches?

From the consistent history prespective there is a ready explanation. In order to apply classical probabilistic concepts within the quantum domain, some regulatory principle must be employed, as otherwise the results will be contradictory. For consistent histories this regulatory principle is the mathematical requirement of consistency. Now it seems plausible that this condition is satisfied for those sequences of macroscopic events which the orthodox interpretation would regard as "results of measurements" (or that it is satisfied "for all practical purposes," which is surely sufficient given that the projections corresponding to macroscipic events can hardly be given an extremely precise definition). The reason is that violations of consistency correspond (speaking somewhat loosely) to the presence of significant quantum interference, and the model calculations of Section 5 and other results referred to in Section 7.2.1 below suggest that such interference is probably suppressed for appropriate sequences of events (excluding, of course, specific "grotesque events," but these are surely not in view in the orthodox approach.)

If the foregoing is correct, the more agnostic type of orthodox approach, that which regards only the "results of measurements" as physically meaningful, is using a regulatory principle which selects out only a tiny subset of consistent histories of $I+S$—but of course an extremely important subset, as it includes those which are crucial for interpreting the macroscopic results of laboratory tests. On this subset it and the consistent history approach give the same probabilities, which explains why (5.51) is the same as (7.1).

But whatever its advantages, this "conservative" approach exacts a heavy price in terms of the conceptual foundations of quantum theory. On the one hand, by making "measurements" a fundamental part of the interpretive scheme, it renders a quantum study of the measurement process itself extremely difficult, and this difficulty is what seems to be ultimately behind the appearance of conscious observers or classical apparatus in discussions to which one feels they should be irrelevant. On the other hand, it does not provide a good way of talking in physical terms about the microscopic world, and as a result one has the odd feeling that quantum mechanics can predict the results of measurements but cannot say precisely what it is that is being measured.

In the less agnostic approach of von Neumann it is possible to say something about what is going on (in a statistical sense) in a closed system between measurements by using the wave function to calculate probabilities in the usual way. The situation is a little odd in that the result of an ideal measurement tells one the state of a system right after the measurement, but nothing about the state just before. In addition the procedure does not make it possible to calculate correlations between states of affairs at several different times inside a closed system.

These limitations can be understood from the consistent history perspective as reflecting what is in effect a regulatory principle which, while admitting a much wider class of events than the more agnostic approach, is still limited to cases in which consistency is "automatically" satisfied: namely, the standard statistical interpretation applied to histories of the form (2.23), where D represents the state of the system just after the last measurement (i.e., t_0 is the time the system became isolated). The peculiar temporal asymmetry is essential, for were there any way in which measurements at the end of the time interval during which the system is closed could give the same sort of information as those at the beginning, the stage would be set for the sort of problem discussed in Section 4, which the orthodox interpretation is not equipped to handle.

In this connection it may be of interest to note that Aharonov *et al.*[9] reached the conclusion that there is equal justification for adopting the opposite temporal convention as von Neumann, with a measurement indicating the state of the system just before instead of just after the measurement, and concluded that this was good grounds for not supposing that a quantum system can be described by a unique wave function, as assumed by quantum orthodoxy. We completely concur with this; see Section 3.3.

In summary, despite very important differences in approach and outlook, the orthodox and consistent history approaches seem to be basically compatible, with the latter providing a number of insights into the successes and conceptual difficulties of the former, while at the same time getting rid of some of the less satisfactory philosophical conclusions which orthodoxy sometimes seems to imply (temporal asymmetry, need for conscious observers), but with which the orthodox themselves have never been very content. But, equally important, the consistent history approach goes well beyond the orthodox interpretation by providing a much more general regulatory principle which makes possible a greatly increased number of precise statements about events in the microscopic domain, in a controlled and sensible way. The price for this is that one must pay attention to the consistency condition and questions of compatibility, as these are *not* automatically fulfilled. But since the need to pay attention to these arises precisely from those features of quantum mechanics which seem the least intuitive in comparison with classical mechanics, the extra effort may actually lead to a clearer understanding of what quantum mechanics is all about.

7.2. Closed Systems

7.2.1. "For All Practical Purposes." One approach to resolving the quantum measurement problem (Section 7.1) is to assert that in suitable cases, in particular when I represents a macroscopic measuring device (or devices), the grotesque wave function, for the combined system $I + S$, which results from a measurement of S by I [e.g., (3.3)] can be collapsed into a corresponding density matrix [e.g., (3.21)] "for all practical purposes": probabilities of subsequent events can be calculated using either the wave function or the density matrix, and the results will be essentially the same. Arguments to this effect have been given by Gottfried,[10] by Cini[11] and his collaborators, and no doubt by others.

We shall not discuss the question as to whether, and if so in what fashion, the proposal under discussion actually resolves the measurement problem. Of greater interest from the standpoint of consistent histories is that arguments of this type can be understood as arguments for the consistency of certain classes of histories. Replacing the wave function ψ by the corresponding density matrix amounts to the substitution

$$|\psi\rangle\langle\psi| \to \sum_{\alpha} E_k^\alpha |\psi\rangle\langle\psi| E_k^\alpha \tag{7.2}$$

where the projections E_k^α, $\alpha = 1, 2, \ldots,$ correspond to the (macroscopically) different physical situations (e.g., which one of several counters actually triggers). But the consistency condition (2.14) with $E_k = 1$ is in effect the assertion that *for the weights of interest* the replacement (7.2) makes no difference.

Consistency depends not on a single event but on an entire sequence of events, and it is easy to exhibit cases [e.g., using appropriate "grotesque" events of the type (3.20)] in which (7.2) is not justified. And, indeed, an essential element in the "for all practical purposes" argument is that the two sides of (7.2) are indistinguishable for "realistic" physical observables, or that actual or reasonably conceivable laboratory experiments cannot detect the "quantum interference" which distinguishes the wave function and the density matrix.

To the extent that these arguments are valid, they justify in some measure the claim made in Section 7.1 that the use of "measurement" in the orthodox interpretation can be thought of as a device for selecting consistent histories (and thus avoiding a discussion of inconsistent histories). They also show that the term "measurement," which is usually not very precisely defined in the orthodox approach, probably carries with it some implicit connotation of "what can be done in practice"—perhaps going well beyond what is practical in the real laboratory, but not *too* far beyond.

On the other hand, the consistency condition in the consistent history approach is a mathematical one, and makes no reference to laboratory practice, now or in the future. The difference which this makes in terms of interpretation is illustrated by the fact that at one point in his discussion (see the footnote of p. 186 of Ref. 10) Gottfried admits that new experimental effects involving quantum interference might make it necessary to qualify some of his assertions and thus the interpretive scheme based on them. No similar concern affects the consistent history approach, for if some clever experimentalist does manage to arrange an event sequence which can detect the quantum interference which distinguishes the two sides of (7.2), then (by definition!) the corresponding history will be inconsistent if it includes the event set $[E_k^\alpha]$, and consequently the consistent history analysis will refuse to answer the question as to whether or not, under these conditions, one of the events E_k^α actually occurred.

7.2.2. Probabilities Obtained Directly from the Wave Function. An alternative approach to the measurement problem is illustrated in a paper by Moldauer,[12] who takes the attitude that the properties of interest for the closed system $I + S$, where I can include various pieces of apparatus interacting with the system S at several successive times, can all be calculated from the wave function ψ for $I + S$, with its time development given by the Schrödinger equation, without any wave function collapse, by using "the rules of quantum mechanics." By the latter he evidently means what we call the "standard statistical interpretation" in Section 2.3 applied to histories of the type

$$D \rightarrow E_1 \rightarrow 1 \tag{7.3}$$

where E_1 may involve "microscopic" properties of S, "macroscopic" properties of I, or both, or in fact any property of $I + S$ which can be represented by a projection operator. Note that all histories of this type are automatically consistent.

Our criticism of Moldauer's approach concerns not what it affirms, but what it leaves unsaid. Consider, as an example, a spin-1/2 particle passing through four successive spin polarization analyzers, measuring polarizations in different directions, at times $t_1 < t_2 < t_3 < t_4$. Suppose that at time $t_0 < t_1$ each of the analyzers is in its "untriggered" state (I_j^0 for analyzer j), and that at time $t_f > t_4$ each analyzer is in a state ($I_j^{\alpha_j}$ for analyzer j) indicating the measured polarization. Given this information, is it true that at time $t_{2.5}$, when the particle was between analyzers 2 and 3, the first two analyzers were in the states $I_1^{\alpha_1}$ and $I_2^{\alpha_2}$, with α_1 and α_2 the same as the corresponding values at t_f, whereas the second two were in the untriggered states I_3^0 and I_4^0?

"Clearly yes," comes the ready reply, "given the appropriate construction and dynamics of the analyzers." But *can this actually be demonstrated from a quantum mechanical analysis* of the closed system of particles plus analyzers, given whatever (quantum not classical) construction and dynamics the theorist prefers? That is a different matter, and in fact the approach which uses only histories of the type (7.3) cannot handle questions of this type. Note that it is not enough to calculate probabilities *separately* for the times $t_{2.5}$ and t_f from the total (uncollapsed) ψ (or equivalently the Heisenberg operator $\hat{D}$) at these two times, for the question involves a *correlation* between the two states of affairs. Nor can the question be simply dismissed as "meaningless," for it is of the sort an experimental physicist might well ask ("Joe, do you think something crazy in the electronics might have triggered number 3 just before the particle got there?"), given that real experiments are seldom as clean as their gedanken counterparts.

We suspect that the typical quantum physicist pressed to answer this question will proceed by collapsing the wave function at $t_{2.5}$ (or at t_1 and again at t_2, etc.) and then using the individual pieces to calculate the future course of events, and hence the way they are correlated with the situation at $t_{2.5}$. With an uneasy conscience if he accepts the orthodox interpretation (for the *total* system of particle plus analyzers is to be thought of as closed during the entire time from t_0 to t_f), an easier conscience if he accepts the arguments of Section 7.2.1 above—and with joyous abandon if he adopts the consistent history perspective (with a promise that the technical issues of proving consistency will be treated in a later publication!). In any case, the main point is that questions of this sort simply cannot be handled by an interpretive scheme which limits itself to (7.3). And as soon as longer histories are considered, or F is not equal to 1, consistency is not automatic and some extension of the standard statistical interpretation must be employed.

The following possibility may occur to some readers and deserves comment. Could it be that the Heisenberg operator corresponding to the situation at interest at $t_{2.5}$ commutes with the Heisenberg operator corresponding to the final state of all the analyzers at t_f (or "almost" commutes, or commutes if it is slightly altered in a way which represents the same macroscopic information, etc.) when the two are referred to a common reference time? If the answer were "yes," one could employ the "generalized standard interpretation," in the terminology of Section 2.3, and consistency would no longer be an issue. Although we have no proof, we think the correct answer is "no," and the same is true if the Heisenberg operator for the initial state replaces that of the final state in the statement of the question. For evidence supporting this, see the remarks at the end of Appendix C.

7.2.3. "Many Worlds." Everett's "relative state" approach[13] to quantum interpretation is based on the idea that the wave function of a closed system ("the universe") developing in time according to the Schrödinger equation provides a correct deterministic description of the system. When in the course of time the wave function becomes grotesque (in our terminology), this is because there has been an actual splitting of the original single macroscopic state-of-affairs into two or more distinctly different macrostates which simultaneously exist (in some sense), though they do not in practice interact with each other. It is claimed by advocates of this position that the usual statistical interpretation of quantum theory can be generated without any probabilistic hypothesis, simply using the Schrödinger dynamics. (We share the suspicions of Ballentine[14] and Benioff[15] that the calculations in support of this thesis actually contain implicit probabilistic hypotheses.)

By contrast, the consistent history interpretation is explicitly probabilistic, has no need to intrepret "grotesque" wave functions in order to discuss ordinary events, and in any case denies that the wave function provide a description of the universe in anything like the sense claimed by Everett. Thus it lacks precisely those features of the Everett interpretation which give rise to its most controversial claim: that the universe is in some sense continually "splitting" into separate worlds with different macroscopic situations in each one.

On the other hand, a certain "splitting" *can* occur within the consistent histories framework, in the following sense. A simple initial state D can lead at some later time to two or more very different macroscopic situations each with a significant probability (see Section 3 for an example). There is a very analogous phenomenon in classical statistical mechanics. The probability distribution in phase space for a system which is initially in a well defined but unstable macrostate may at some later time "split" into one which represents several very different macrostates, each with a significant probability. However, this phenomenon is interpreted either by saying that a single system starting in the initial macrostate will at a later time be in *one* of the macrostates which has significant probability (but the theorist, owing to his ignorance, cannot say which), or that in an ensemble of systems corresponding to the original probability distribution, some will later be found in one macrostate and some in another. In neither case does one think of an *individual* system as somehow "split" between different macrostates.

We grant that quantum probabilities do not behave in all respects like their classical counterparts. But given that the consistency condition selects families of histories whose probabilities satisfy the classical rules (in a mathematical sense), it seems most natural to interpret the "splitting" which takes place inside a consistent family—this seems to be the situation which Everett has in mind—in terms of the classical analogy, or at least there seems to be no reason *not* to do so.

APPENDIX A: WEIGHTS W AS PROBABILITIES FOR CONSISTENT HISTORIES

In what follows we restrict our attention to a single family of histories of the form (2.6), with D, F, and the different event sets held fixed. There are then a total of

$$\prod_{k=1}^{n} [2^{M_k} - 1] \tag{A1}$$

separate histories in the family, including the trivial history in which every E_k is equal to 1. Let us call those histories in which for each k E_k is one of the E_k^α [see (2.4)] "elementary" histories, and the other cases "compound" histories.

When dealing with classical probabilities (that is, the subject discussed in the usual textbook on probability theory, Feller[16] for example), the following rule applies. Let A and B be two mutually exclusive events [using "event" in the probabilistic sense, (see p. 8 of Ref. 16), which is more general than the sense of Section 2.1] and C any other event. Then the probability of the event $(A \vee B) \wedge C$, "A or B, and C," is the sum of the probabilities of $A \wedge C$ and $B \wedge C$. In the quantum case the E_k^α for fixed k and different α corresponds to mutually exclusive events, as the product (2.5) vanishes for $\alpha \neq \beta$. Thus if the classical rule just discussed is to apply to the histories making up the family we are discussing, we are naturally led to the demand that

$$P(E_1 \wedge \cdots E_k \wedge \cdots E_n) = \sum_{\alpha}{}' P(E_1 \wedge \cdots E_k^\alpha \wedge \cdots E_n) \tag{A2}$$

where P stands for the probability of the history, and just as in (2.14) and (2.15), the sum on the right is over those α corresponding to the projections which make up E_k on the left.

Provided (A2) holds for all k and all possible choices of $E_1, E_2, \ldots, E_n$, it is equivalent to the demand that

$$P(E_1 \wedge \cdots E_n) = \sum_{\alpha_1}{}' \sum_{\alpha_2}{}' \cdots \sum_{\alpha_n}{}' P(E_1^{\alpha_1} \wedge E_2^{\alpha_2} \wedge \cdots E_n^{\alpha_n}) \tag{A3}$$

hold for all possible choices of $E_1, E_2, \ldots, E_n$, with the sum over α_j on the right side restricted to those projections which make up E_j on the left. That is to say, a compound history may be thought of as composed of a set of mutually exclusive elementary histories, and its probability is a sum of the probabilities of the latter. A *consistent* family of histories is then one for which we can make the identification

$$P(E_1 \wedge \cdots E_n) = W(E_1 \wedge \cdots E_n \,|\, D \wedge F) \tag{A4}$$

where W is given by (2.12), for all histories belonging to the family, and have (A2), which means (2.14), satisfied. The fact that (A3) holds for this family is what justifies treating the corresponding probabilities by the classical rules when computing conditional probabilities, etc.

An alternative approach might be to make the identification (A4) for elementary histories but not for compound histories, and then define the probability of the latter using (A3). Provided the sum of the probabilities of all the elementary histories is 1, which will automatically be the case if $F = 1$ (or $D = 1$), the resulting structure can, of course, be treated by classical rules. [And one can, if necessary, introduce a constant factor on the right side of (A4) to ensure that probabilities of the elementary histories sum to one.] However, this approach has the disadvantage that unless the family is consistent, the identification (A4) will not hold for all of the compound histories. In particular, the probability of, say,

$$E_1 \wedge 1 \wedge \cdots \wedge 1 \tag{A5}$$

which would normally be interpreted simply as "E_1 occurred at time t_1," will in general depend on the particular (inconsistent) family of histories in which this event is embedded.

Checking the consistency condition (2.14) is a lengthy task for any but the shortest and simplest histories. This task is made slightly easier using the alternative (2.19), which we now derive. Let P and Q be any two bounded Hermitian operators, at least one of which is of trace class, and for B and C any two bounded operators, define

$$\begin{aligned}\langle B, C\rangle &= \mathrm{Re}\,\mathrm{Tr}\left[PB^{\dagger}QC\right]\\ &= \tfrac{1}{2}\mathrm{Tr}\left[PB^{\dagger}QC\right] + \tfrac{1}{2}\mathrm{Tr}\left[PC^{\dagger}QB\right]\end{aligned} \tag{A6}$$

where $B^{\dagger}$ is the Hermitian conjugate of B. It is at once evident that

$$\langle B, C\rangle = \langle C, B\rangle \tag{A7}$$

$$\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle \tag{A8}$$

Now let

$$\begin{aligned} P &= \hat{E}_{k+1}\hat{E}_{k+2} \cdots \hat{E}_n\hat{F}\hat{E}_n \cdots \hat{E}_{k+2}\hat{E}_{k+1}\\ Q &= \hat{E}_{k-1}\hat{E}_{k-2} \cdots \hat{E}_1\hat{D}\hat{E}_1 \cdots \hat{E}_{k-2}\hat{E}_{k-1}\end{aligned} \tag{A9}$$

and note that (2.14) is equivalent to

$$\langle \hat{E}_k, \hat{E}_k\rangle = \sum_{\alpha}{}' \langle \hat{E}_k^{\alpha}, \hat{E}_k^{\alpha}\rangle \tag{A10}$$

Upon writing out the left side using (2.15), (A7), and (A8), one sees that

(A10) will be true provided

$$\langle \hat{E}_k^\alpha, \hat{E}_k^\beta \rangle = 0 \tag{A11}$$

for all $\alpha < \beta$. On the other hand, if (A11) does not hold for some $\alpha < \beta$, (A10) will be violated for E_k equal to $E_k^\alpha + E_k^\beta$. Of course (A11) is the same as (2.19).

APPENDIX B: ARGUMENT THAT $\hat{A}_1$ (SECTION 3) DOES NOT COMMUTE WITH $\hat{D}$ OR $\hat{K}_a^+$

If we use t_1 as the reference time, $\hat{A}_1 = A_1$ and $\hat{D} = D_1$, with D_1 defined in (3.8). But then, using (3.4), we see that

$$A_1 D_1 \Psi_1 = \Psi_1^a/\sqrt{2}, \qquad D_1 A_1 \Psi_1 = \Psi_1/2 \tag{B1}$$

so that the two operators do not commute.

The argument that $\hat{A}_1$ does not commute with $\hat{K}_a^+$ is more complicated. Consider a wave function (*not* produced by scattering!)

$$\bar{\Psi} = \bar{\psi}_1 C_a C_b \tag{B2}$$

at time t_1, where $\bar{\psi}_1$ is a wavepacket located in a small region centered on the boundary of A_1, chosen so that as time advances, it will spread in different directions with a pattern yielding destructive interference (a low amplitude) in that portion moving towards the counter C_a. By an appropriate choice of the initial wavepacket we can also arrange that $A_1\bar{\psi}_1$ does not suffer that same destructive interference and will have a much larger amplitude for that portion of the wave moving toward C_a. Consequently this second wavepacket leads to a larger probability for counter a in the triggered state at time t_2:

$$\begin{aligned} &\langle \bar{\Psi} | A_1 U(t_1, t_2) K_a^+ U(t_2, t_1) A_1 | \bar{\Psi} \rangle \\ &\quad > \langle \bar{\Psi} | U(t_1, t_2) K_a^+ U(t_2, t_1) | \bar{\Psi} \rangle \end{aligned} \tag{B3}$$

However, choosing the reference time t_r as t_1, we can rewrite this as

$$\langle \bar{\Psi} | \hat{A}_1 \hat{K}_a^+ \hat{A}_1 | \bar{\Psi} \rangle > \langle \bar{\Psi} | \hat{K}_a^+ | \bar{\Psi} \rangle \tag{B4}$$

On the other hand, if $\hat{A}_1$ and $\hat{K}_a^+$ were to commute we would have the result, with $\hat{A}_1' = 1 - \hat{A}_1$:

$$\begin{aligned} \langle \bar{\Psi} | \hat{K}_a^+ | \bar{\Psi} \rangle &= \langle \bar{\Psi} | \hat{K}_a^+ \hat{A}_1 | \bar{\Psi} \rangle + \langle \bar{\Psi} | \hat{K}_a^+ \hat{A}_1' | \bar{\Psi} \rangle \\ &= \langle \bar{\Psi} | \hat{A}_1 \hat{K}_a^+ \hat{A}_1 | \bar{\Psi} \rangle + \langle \bar{\Psi} | \hat{A}_1' \hat{K}_a^+ \hat{A}_1' | \bar{\Psi} \rangle \\ &\geqslant \langle \bar{\Psi} | \hat{A}_1 \hat{K}_a^+ \hat{A}_1 | \bar{\Psi} \rangle \end{aligned} \tag{B5}$$

in contradiction with (B4).

Note that this argument does not depend on precisely which operator is employed for K_a^+; any reasonable choice will do. On the other hand there is clearly something rather artificial about a construction which depends on the precise boundary of the region A_1. The moral to be drawn from this, in our opinion, is that the demand for commutativity of the Heisenberg operators, referred to the same reference time, for two events taking place at different times is neither physically sensible nor mathematically simple. The consistency condition appears to be preferable in both respects.

APPENDIX C: DETAILS OF CONSISTENCY AND PROBABILITY CALCULATIONS FOR SECTION 4

It is convenient to use t_2 as the reference time t_r, and the following abbreviations:

$$\begin{aligned} |1\rangle &= \gamma Z^+X^+, \qquad |2\rangle = \delta Z^+X^- \\ |3\rangle &= \gamma Z^-X^+, \qquad |4\rangle = \delta Z^-X^- \end{aligned} \tag{C1}$$

for the states, mutually orthogonal and normalized, which occur in (4.6):

$$|\Psi_2\rangle = \tfrac{1}{2}\{|1\rangle + |2\rangle + |3\rangle - |4\rangle\} \tag{C2}$$

Note that as D corresponds to Ψ_0 at t_0,

$$\hat{D} = |\Psi_2\rangle\langle\Psi_2| \tag{C3}$$

By using (4.3) and (4.1), one can find the counterparts of the states (C1) at time t_1 and use these to calculate matrix elements of $\hat{\mathrm{A}}_1$ and $\hat{\Gamma}_1$. For example,

$$\begin{aligned} \hat{\mathrm{A}}_1|1\rangle &= U(t_2,t)\mathrm{A}_1U(t_1t_2)|1\rangle = U(t_2,t_1)[\mathrm{A}_1\gamma Z^+X] \\ &= U(t_2,t_1)[\alpha Z^+X]/\sqrt{2} = \tfrac{1}{2}\{|1\rangle + |2\rangle\} \end{aligned} \tag{C4}$$

The result is

$$\hat{\mathrm{A}}_1 = \tfrac{1}{2}1 + \tfrac{1}{2}\{|1\rangle\langle 2| + |2\rangle\langle 1| + |3\rangle\langle 4| + |4\rangle\langle 3|\} \tag{C5}$$

where

$$1 = \sum_{j=1}^{4} |j\rangle\langle j| \tag{C6}$$

and

$$\hat{\Gamma}_1 = |1\rangle\langle 1| + |3\rangle\langle 3| \tag{C7}$$

while, of course,

$$\hat{F} = F = |1\rangle\langle 1| \tag{C8}$$

Since $\hat{\Gamma}_1$ and $\hat{F}$ commute, the consistency of (4.8) is automatic. To check the consistency of (4.7), we evaluate (2.20), noting that

$$\hat{A}_1'\hat{F}\hat{A}_1 = \tfrac{1}{4}\{|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| - |2\rangle\langle 1|\} \tag{C9}$$

and thus

$$\mathrm{Tr}[\hat{A}_1\hat{D}\hat{A}_1'\hat{F}] = \langle\Psi_2|\hat{A}_1'\hat{F}\hat{A}_1|\Psi_2\rangle = 0 \tag{C10}$$

Hence the conditional probabilities (4.9) and (4.10) are equal to the corresponding weights W in (2.12). The latter can be expressed in terms of

$$\mathrm{Tr}[\hat{D}\hat{F}] = 1/4 = \mathrm{Tr}[\hat{\Gamma}_1\hat{D}\hat{\Gamma}_1\hat{F}] = \mathrm{Tr}[\hat{A}_1\hat{D}\hat{A}_1\hat{F}] \tag{C11}$$

where the traces are evaluated in the same manner as in the consistency check.

In considering the histories (4.14) and (4.15), we note that $\hat{A}_{1.1} = \hat{A}_1$ and $\hat{\Gamma}_{1.1} = \hat{\Gamma}_1$. What distinguishes the two cases is the order of the operators inside the traces. In checking the consistency of (4.14) using (2.20), we note that the cases in which $\hat{G}_1$ or $\hat{G}_{1.1}$ are equal to 1 correspond with the histories (4.8) and (4.7), whose consistency has already been checked. The remaining cases are easily taken care of by noting that

$$\hat{\Gamma}_{1.1}'\hat{F} = 0 = \hat{F}\hat{\Gamma}_{1.1}', \qquad \hat{\Gamma}_{1.1}\hat{F} = \hat{F} = \hat{F}\hat{\Gamma}_{1.1} \tag{C12}$$

which also makes it evident that the probabilities in (4.9) and (4.16) are the same. On the other hand,

$$\mathrm{Tr}[\hat{A}_{1.1}\hat{\Gamma}_1\hat{D}\hat{\Gamma}_1\hat{A}_{1.1}'\hat{F}] = 1/16 \tag{C13}$$

is nonzero and real, so (4.15) cannot be consistent.

The above calculations using a four-dimensional vector space are obviously a drastic oversimplification in terms of what should in principle be employed for "realistic" apparatus. We do not, on the other hand, think that they are misleading. In this connection, we note one respect in which they can easily be improved. As pointed out in Section 5, an idealized measuring instrument needs a minimum of two states. Since we are considering two instruments plus a spin-1/2 particle, the vector space should be eight dimensional instead of four, which is to say we should supplement (C1) with

$$\begin{aligned} |5\rangle &= \gamma Z^+X^-, &\qquad |6\rangle &= \delta Z^+X^+ \\ |7\rangle &= \gamma Z^-X^-, &\qquad |8\rangle &= \delta Z^-X^+ \end{aligned} \tag{C14}$$

and augment (4.3) by adding the transformations

$$\gamma\tilde{X} \to \gamma X^-, \qquad \delta\tilde{X} \to \delta X^+ \tag{C15}$$

where X and $\tilde{X}$ are the two possible states of the S_x analyzer before the particle arrives. Similarly, to (4.2) one should add the transformations

$$\alpha\tilde{Z} \to \alpha Z^-, \qquad \beta\tilde{Z} \to \beta Z^+ \tag{C16}$$

The net result of repeating the previous calculations in this enlarged space is that one gets *precisely* the same results. The reason is that $\hat{\mathrm{A}}_1$, $\hat{\Gamma}_1$, $\hat{D}$, and $\hat{F}$ have no matrix elements connecting the subspace spanned by (C1) and that spanned by (C14), and the latter is annihilated by $\hat{D}$, which enters all of the traces considered in these histories.

It is also possible, using this eight-dimensional space, to examine the Heisenberg operators corresponding to events which make reference only to states of the analyzers and not to the spin of the particle. This is a useful exercise if for no other reason than that one's intuition about the behavior of such operators is often defective. Among the results which emerge—we leave the argument as an exercise to the reader—is that the Heisenberg operators for ZX at time t_0 and Z^+X^+ at time t_2 do *not* commute when referred to a common reference time (t_2 is a convenient choice).

APPENDIX D: DERIVATION OF (5.17) AND (5.42)

We begin with (5.17). It is convenient to write the argument of the trace on the left side in the "long" form (2.11), which is, letting $t_i = t_f$,

$$\begin{aligned} &J_i^\gamma U_c(t_f, t_n) E_n U_c(t_n, t_{n-1}) \\ &\quad \ldots E_{m+1} U_c(t_{m+1}, t_m) E_m \ldots U_c(t_1, t_0)[d \otimes D] U_c(t_0, t_1) \\ &\quad \ldots G_m U_c(t_m, t_{m+1}) G_{m+1} \ldots G_n U_c(t_n, t_f)[1 \otimes F] \end{aligned} \tag{D1}$$

In view of the time ordering and of (5.18), all the U_c can be replaced by U except for the cases where one argument is t_m and one is t_{m+1}, where (5.5) and (5.6) must be used. With the help of (5.7), (5.8), and (5.15), and after shifting the operators on I to the left side, we obtain

$$\begin{aligned} &\sum_\alpha \sum_\beta (r^\gamma s^\alpha d s^{-\beta}) \\ &\quad \otimes \big[U(t_f, t_n) E_n \ldots E_{m+1} U(t_{m+1}, \tau) A_\tau^\alpha U(\tau, t_m) E_m \\ &\quad \ldots U(t_1, t_0) D U(t_0, t_1) \ldots G_m U(t_m, \tau) A_\tau^\beta U(\tau, t_{m+1}) G_{m+1} \\ &\quad \ldots G_n U(t_n, t_f) F \big] \end{aligned} \tag{D2}$$

Now the trace Tr_c of each summand in (D2) can be written as a product of the trace tr over I of the operator product inside parentheses multiplied by the trace Tr over S of the operator product inside square brackets. Using the definitions (5.9), (5.13), and (5.14), together with the orthonormality of the ξ^α, one obtains

$$\begin{aligned}\operatorname{tr}(r^\gamma s^\alpha d s^{-\beta}) &= \langle \xi^\gamma | s^\alpha | \xi^0 \rangle \langle \xi^0 | s^{-\beta} | \xi^\gamma \rangle \\ &= \delta_{\beta\gamma}\delta_{\alpha\gamma} \end{aligned} \tag{D3}$$

and inserting this in the double sum yields the "long form" of the trace on the right side of (5.17).

The same approach will work for (5.42). When the argument of Tr_c is written in the "long form" and reexpressed using (5.30) to (5.35), the result which corresponds to (D2) is

$$\begin{aligned}&\sum_{\beta_1}\sum_{\beta_2}\cdots\sum_{\beta_\nu}\sum_{\gamma_1}\cdots\sum_{\gamma_\nu}\left(r_1^{\alpha_1} r_2^{\alpha_2}\ldots r_\nu^{\alpha_\nu} s_\nu^{\beta_\nu}\ldots s_2^{\beta_2} s_1^{\beta_1} d s_1^{-\gamma_1} s_2^{-\gamma_2}\ldots s_\nu^{-\gamma_\nu}\right)\\ &\qquad \otimes\left[\ldots A_\nu^{\beta_\nu}\ldots E_k\ldots A_1^{\beta_1}\ldots D\ldots A_1^{\gamma_1}\ldots G_k\ldots A_\nu^{\gamma_\nu}\ldots F\right]\end{aligned} \tag{D4}$$

where the operator product in square brackets is similar to that in (D2), except that there are ν A's on each side of D, each inserted at the position which maintains the proper time sequence, with (of course) the appropriate U operators. The trace Tr_c of each summand in (D4) factors into a product of the trace over I of the operator product in parentheses and the trace over S of the operator product in square brackets; the former in turn factors into traces over the individual I_μ of the form

$$\operatorname{tr}_\mu\left(r_\mu^{\alpha_\mu} s_\mu^{\beta_\mu} r_\mu^0 s_\mu^{-\gamma_\mu}\right) = \delta_{\alpha_\mu\beta_\mu}\delta_{\alpha_\mu\gamma_\mu} \tag{D5}$$

Inserting these in the multiple sum yields the right side of (5.42).

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REFERENCES

1. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), translated from *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
2. A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**:777 (1935).
3. M. Jammer, *The Philosophy of Quantum Mechanics* (John Wiley & Sons, New York, 1974).
4. E. P. Wigner, *Am. J. Phys.* **31**:6 (1963).
5. F. London and E. Bauer, *La Théorie de l'Observation en Mécanique Quantique* (Hermann, Paris, 1939).
6. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, London, 1958).
7. A. Messiah, *Mécanique Quantique* (Dunod, Paris, 1965); English translation: *Quantum Mechanics* (North-Holland, Amsterdam, 1970).
8. L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968).
9. Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, *Phys. Rev.* **134**:B1410 (1964).
10. K. Gottfried, *Quantum Mechanics*. Vol. I: *Fundamentals* (W. A. Benjamin, New York, 1966), Chap. IV.
11. M. Cini, *Nuovo Cim.* **73B**:27 (1983); M. Cini *et al.*, *Found. Phys.* **9**:479 (1979).
12. P. A. Moldauer, *Phys. Rev. D* **5**:1028 (1972).
13. H. Everett III, *Rev. Mod. Phys.* **29**:454 (1957); B. S. DeWitt in *Battelle Rencontres*: 1967 *Lectures in Mathematics and Physics*, C. M. DeWitt and J. A. Wheeler, eds. (W. A. Benjamin, New York, 1968); B. S. DeWitt and N. Graham, eds., *The Many Worlds Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1973).
14. L. E. Ballentine, *Found. Phys.* **3**:229 (1973).
15. P. Benioff, *Found. Phys.* **8**:709 (1978).
16. W. Feller, *An Introduction to Probability Theory and its Applications*, 3rd ed. (John Wiley and Sons, New York, 1968).